LOG-MAJORIZATION OF THE MODULI OF THE EIGENVALUES OF A MATRIX POLYNOMIAL BY TROPICAL ROOTS

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Abstract. We show that the sequence of moduli of the eigenvalues of a matrix polynomial is log-majorized, up to universal constants, by a sequence of "tropical roots" depending only on the norms of the matrix coefficients. These tropical roots are the non-differentiability points of an auxiliary tropical polynomial, or equivalently, the opposites of the slopes of its Newton polygon. This extends to the case of matrix polynomials some bounds obtained by Hadamard, Ostrowski and Pólya for the roots of scalar polynomials. We also obtain new bounds in the scalar case, which are accurate for "fewnomials" or when the tropical roots are well separated.

Key words. Matrix polynomial, Tropical algebra, Majorization of eigenvalues, Tropical roots, Roots of polynomial, Bound of Pólya.

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1. Introduction. Let $p(x) = \sum_{j=0}^{n} a_j x^j$, $a_j \in \mathbb{C}$ be a polynomial of degree n in a complex variable x. Let $\zeta_1, \ldots, \zeta_n$ denote the roots of $p(x)$ arranged by non-decreasing modulus (i.e., $|\zeta_1| \leqslant \ldots \leqslant |\zeta_n|$). We associate to p the *tropical polynomial* $\mathsf{t}\,p(x)$, defined for all nonnegative numbers x by

$$\mathsf{t}\,p(x) := \max_{0 \leqslant j \leqslant n} |a_j| x^j \enspace .$$

The *tropical roots* of $\mathsf{t}\,p$, $\alpha_1, \ldots, \alpha_n$, ordered by non-decreasing value (i.e., $\alpha_1 \leqslant \ldots \leqslant \alpha_n$), are defined as the non-differentiability points of the function $\mathsf{t}\,p$, counted with certain multiplicities. They coincide with the exponential of the opposite of the slopes of the edges of a Newton polygon, defined by Hadamard [Had93] and Ostrowski [Ost40a, Ost40b] as the upper boundary of the convex hull of the set of points $\{(j, \log|a_j|) \mid 0 \leqslant j \leqslant n\}$. The logarithms of these roots were called the *inclinaisons numériques* by Ostrowski. One interest of these roots is that they can be easily computed (linear number of arithmetic operations and comparisons). See Section 3 below for details.

Hadamard was probably the first to prove a log-majorization type inequality for the modulus of the roots of a scalar polynomial by using what we call today the tropical roots. His result (page 201 of [Had93], third inequality) can be restated as follows in tropical terms:

$$\frac{|\zeta_1 \zeta_2 \ldots \zeta_k|}{\alpha_1 \cdots \alpha_k} \geqslant \frac{1}{k+1} \enspace . \tag{1.1}$$

This bound, proved in passing in a memoir devoted to the Riemann zeta function, remained apparently not so well known. In particular, the special case $|\zeta_1|/\alpha_1 \geqslant 1/2$

is equivalent to the homogeneous form of the classical bound of Cauchy, established later on by Fujiwara [Fuj16], and a weaker inequality, with α_1^k at the denominator instead of $\alpha_1 \cdots \alpha_k$, appeared later on in the work of Specht [Spe38].

Ostrowski proved several bounds on the roots of a polynomial in his work on the method of Graeffe [Ost40a, Ost40b], in which he used again the Newton polygon considered by Hadamard. In particular, he obtained the following upper bound (see [Ost40a, §7]),

$$\frac{|\zeta_1 \zeta_2 \cdots \zeta_k|}{\alpha_1 \cdots \alpha_k} \leqslant \binom{n}{k} \;, \tag{1.2}$$

which can be thought of as a generalization of a "reverse" of the Cauchy inequality due to Birkhoff [Bir15] (corresponding to the case $k = 1$ in (1.2)). He also gave a different proof of a variant of (1.1), with the constant $1/(2k)$ instead of $1/(k+1)$, and reported a private communication of Pólya, leading to a tighter constant

$$\frac{|\zeta_1 \zeta_2 \cdots \zeta_k|}{\alpha_1 \cdots \alpha_k} \geqslant \frac{1}{\sqrt{\mathcal{E}(k)(k+1)}} \;, \tag{1.3}$$

with

$$\mathcal{E}(k) := \left(\frac{k+1}{k} \right)^k < e \;. \tag{1.4}$$

In this paper, we generalize the bounds of Hadamard, Ostrowski, and Pólya, to the case of a matrix polynomial

$$P(\lambda) = A_0 + A_1 \lambda + \cdots + A_d \lambda^d, \quad A_j \in \mathbb{C}^{n \times n} \quad j = 0, \ldots d \;.$$

We now associate to the matrix polynomial P the tropical polynomial

$$\mathsf{t}p(x) := \max_{0 \leqslant j \leqslant d} \|A_j\| x^j \;, \tag{1.5}$$

where $\| \cdot \|$ is a norm on the space of matrices, and show that the moduli of the roots $\zeta_1, \ldots, \zeta_{nd}$ of P can still be controlled in terms of the tropical roots $\alpha_1, \ldots, \alpha_n$ of $\mathsf{t}p$. Our results give bounds on the ratio $|\zeta_1 \ldots \zeta_{nk}|/(\alpha_1 \ldots \alpha_k)^n$, which extend and refine the above bounds. In particular, in Theorem 2.4, we extend the lower bound (1.3) of Pólya to the matrix polynomial case, and in Theorem 2.12, we extend the upper bound (1.2) of Ostrowski. Moreover, we obtain other lower bounds that are new even in the case of scalar polynomials. In particular, in Theorem 2.1, we obtain a lower bound which may be tighter for "fewnomials". In Theorems 2.5 and 2.6, we obtain general lower bounds, which extend the bound of Polya and its extension to the matrix case, and which may be much tighter when the tropical roots are sufficiently separated. Then, all together our results show that the tropical roots give tight estimates of the moduli of the eigenvalues if the tropical roots are sufficiently separated and if certain matrices are sufficiently well conditioned.

The results of the present paper combine ideas from max-plus algebra and tropical geometry, and numerical linear algebra. In [ABG98, ABG04, ABG05], Akian, Bapat, and Gaubert studied the eigenvalues and eigenvectors of matrices and matrix polynomials which entries (the A_j) are functions, for instance Puiseux series, of a (perturbation) parameter. It is shown there that the leading exponents of the Puiseux

series representing the different eigenvalues (resp. eigenvectors) coincide, under some genericity conditions, with certain "tropical eigenvalues" (resp. eigenvectors). This can be interpreted in the light of tropical geometry (see [Vir01, IMS07, RGST05] for introductions), where one defines the amoeba of an algebraic variety $V \subset \mathbb{K}^n$ over a field $\mathbb{K}$, equipped with a valuation ν, as the image of V by the map which applies the valuation ν entrywise. When ν is non-archimedean, in particular for the usual valuation (leading exponent) on the field of Puiseux series, Kapranov theorem [EKL06] characterizes the amoeba of a hypersurface $q = 0$ as the set of "roots" (nondifferentiability points) of the tropical polynomial function with coefficients equal to the valuations of the coefficients of q. Some of the results of [ABG98, ABG04, ABG05] may be interpreted in terms of nonarchimedean amoebas, by considering the hypersurface defined by the characteristic equation (the eigenvalue λ and the entries of the matrices A_i appearing in the matrix polynomial being thought of as variables).

Results of tropical geometry suggest that amoebas obtained by taking the valuation $z \mapsto \log|z|$ on the field of complex numbers $\mathbb{C}$ can be approximated by non-archimedean amoebas. Hence, somehow similar results to the ones of [ABG98, ABG04, ABG05] can be expected to hold when using this latter valuation, instead of the usual non-archimedean valuation, with equalities replaced by bounds, due to the approximation nature. Indeed, the above bounds of Hadamard, Ostrowski, and Pólya can be seen as the (archimedean) analogue of the results of [ABG04] in the particular one dimensional case ($n = 1$). Moreover, in the case of a matrix with non-negative coefficients, Friedland [Fri86] established a bound for its spectral radius (or its Perron eigenvalue) in terms of a certain maximal circuit mean, which can be interpreted as a bound of the maximal eigenvalue of A by the maximal tropical eigenvalue of the valuation of A [ABG06].

Here we rather replaced the valuation on $\mathbb{C}$ by the "valuation" $A \mapsto \log\|A\|$ on the ring of $n \times n$ matrices over $\mathbb{C}$, which leads to (1.5). The idea of using this valuation was inspired by several works in numerical linear algebra, which suggested that the information of the norms is relevant. For instance, Higham and Tisseur [HT03] extended to matrix polynomials the bound of Cauchy (related to the special case $k = 1$ in the Hadamard-Ostrowski-Pólya inequality), by using the norms of $A_0^{-1}A_j$ and $A_d^{-1}A_j$. Fan, Lin and Van Dooren [FLVD04] introduced a scaling based on the norms of the matrix coefficients of a matrix quadratic polynomial. In [GS09], the tropical polynomial of (1.5) was initially introduced to refine the results of [FLVD04]. It was shown there (see also [Sha11]) that the tropical roots of $\mathsf{t}p$ can be used to perform scalings allowing one to improve the backward stability of eigenvalue computations for a matrix quadratic polynomial P ($d = 2$). Some bounds on the modulus of the eigenvalues of P, involving the tropical roots of $\mathsf{t}p$, also appeared in [GS09] in the case in which $d = 2$, and in [Sha11] for the smallest and largest tropical roots when $d \geqslant 2$, which may be seen as particular cases of the lower bounds of the present paper.

Let us finally point out some further related works. Bini used in [Bin96] what we call the tropical roots (from the Newton polygon technique) to initialize the Aberth method of computation of the roots of a scalar polynomial. Also, Malajovich and Zubelli applied Ostrowski's analysis to effective root solving [MZ01]. Finally, Bini, Noferini, and Sharify used recently the tropical roots of (1.5) to generalize Pellet's theorem [BNS12].

The paper is organized as follows. The main results are stated in Section 2. In Section 3, we recall the construction of the Newton polygon and give details of the

definition of the tropical roots. In Section 4, we establish the main lower bounds in the case of scalar polynomials, since the proof arguments are more transparent in this case. The extension to matrices, and the other results of Section 2, are proved in Section 5. In Section 6 we provide examples showing the tightness of the bounds.

2. Statement of the main results. The inequalities we present here depend on the choice of a norm $\|\cdot\|$ on the space of matrices. We shall in particular consider the following assumption:

(A1) $|\det A| \leqslant \|A\|^n$, for all $A \in \mathbb{C}^{n \times n}$.

Moreover, the *normalized Frobenius norm,*

$$\|A\|_{\mathrm{nF}} := \left(\frac{1}{n} \sum_{i,j=1}^{n} |A_{ij}|^2 \right)^{\frac{1}{2}} , \quad \forall A \in \mathbb{C}^{n \times n} ,$$

will allow us to generalize Pólya's inequality. Therefore, the next assumption will also be considered:

(A2) There exists $Q, Q' \in \mathbb{C}^{n \times n}$, such that $\det Q = \det Q' = 1$ and $\|QAQ'\|_{\mathrm{nF}} \leqslant \|A\|$, for all $A \in \mathbb{C}^{n \times n}$.

We shall see in Section 5 that Assumption (A2) implies Assumption (A1), and that a number of commonly used norms satisfy Assumption (A1) or Assumption (A2).

Our first theorem provides a lower bound involving the number of nonzero coefficients of a matrix polynomial. This bound is specially useful when the matrix polynomial is a "fewnomial". Recall that the definition of the tropical roots can be found in Section 3, see also Figure 2.1.

THEOREM 2.1 (Bounds involving the number of nonzero coefficients). *Consider a matrix polynomial P with degree d:*

$$P(\lambda) = A_0 + A_1\lambda + \cdots + A_d\lambda^d, \quad A_j \in \mathbb{C}^{n \times n}, \quad 0 \leqslant j \leqslant d ,$$

and let $\zeta_1, \ldots, \zeta_{nd}$ denote its eigenvalues, arranged by non-decreasing modulus. Assume that $\|\cdot\|$ is any norm on the space of matrices satisfying (A1), that $\det A_0 \neq 0$ and let $c = \frac{|\det A_0|}{\|A_0\|^n}$. Let $\alpha_1, \ldots, \alpha_d$ be the tropical roots of the following tropical polynomial:

$$\mathsf{t}p(x) = \max_{0 \leqslant j \leqslant d} (\|A_j\|x^j) ,$$

arranged in non-decreasing order. Also let $\mathrm{mon}\,P$ denote the number of nonzero monomials of P. Then, for all $1 \leqslant k \leqslant d$, we have

$$\frac{|\zeta_1 \ldots \zeta_{nk}|}{(\alpha_1 \ldots \alpha_k)^n} \geqslant c(L_k)^n , \tag{2.1}$$

with

$$L_k = \frac{1}{\mathrm{mon}\,P} . \tag{2.2}$$

Moreover, when $\|\cdot\|$ satisfies (A2), the constant L_k can be replaced by the greater one:

$$L_k = \frac{1}{\sqrt{\mathrm{mon}\,P}} . \tag{2.3}$$

REMARK 2.2. *When $n = 1$ (thus the matrices $A_0, \ldots, A_d$ are scalars), any norm is proportional to the normalized Frobenius norm which is nothing but the modulus map $|\cdot|$, satisfies* (A2), *and for which $c = 1$. Therefore the tropical roots of $\mathtt{t}p$ are the same for all norms, the best possible inequality* (2.1) *is*

$$\frac{|\zeta_1 \ldots \zeta_k|}{\alpha_1 \ldots \alpha_k} \geqslant L_k \ , \tag{2.4}$$

and this inequality holds with L_k as in (2.3).

REMARK 2.3. *If a norm $\|\cdot\|$ on the space of matrices satisfies* (A1) *but not* (A2), *we can still obtain a bound of the form* (2.3) *by changing the constant c. Indeed, by the equivalence between norms on $\mathbb{C}^{n \times n}$, for any norm $\|\cdot\|$ on $\mathbb{C}^{n \times n}$, there exists a constant $\eta > 0$ (which depend on n) such that $\|A\|_{\mathrm{nF}} \leqslant \eta \|A\|$ for all $A \in \mathbb{C}^{n \times n}$. Then, the norm obtained by multiplying $\|\cdot\|$ by η satisfies* (A2), *hence* (2.3). *Since the tropical roots of $\mathtt{t}p$ and $\eta \mathtt{t}p$ are the same, we deduce that* (2.3) *holds for $\|\cdot\|$ with c replaced by c/η^n. However, if $\|\cdot\|$ satisfies* (A1) *but not* (A2), *then $\eta > 1$, so that Inequality* (2.3) *with c/η^n may be weaker than Inequality* (2.2): *this is indeed the case if and only if $\eta > \sqrt{\operatorname{mon} P}$. The same type of conclusions can be obtained for the lower bounds that are stated in the next theorems.*

The following theorem provides a lower bound generalizing the lower bound of Pólya to the matrix case (since, as said in Remark 2.2, when $n = 1$, the modulus map is a norm satisfying (A2), and for which Inequality (2.1) reduces to (2.4)). Up to the constant c, the following bounds are independent of the coefficients of the matrix polynomial P.

THEOREM 2.4 (Universal bound). *Let $A_0, \ldots, A_d, P, \zeta_1, \ldots, \zeta_{nd}, \|\cdot\|, c$, and $\alpha_1, \ldots, \alpha_d$ be as in the first part of Theorem 2.1. Then, for all $1 \leqslant k \leqslant d$, Inequality* (2.1) *holds with L_k defined as follows:*

$$L_k = \frac{1}{\mathcal{E}(k)(k+1)} \ , \tag{2.5}$$

where $\mathcal{E}$ is defined as in (1.4). *Moreover, when $\|\cdot\|$ satisfies* (A2), *Inequality* (2.1) *holds with the greater constant:*

$$L_k = \frac{1}{\sqrt{\mathcal{E}(k)(k+1)}} \ . \tag{2.6}$$

The lower bound of Pólya, and its matrix version given above, are tight only for k small. By symmetry, one can obtain a tight lower bound when k is close to d (this was already noted by Ostrowski in the scalar case [Ost40a]). Theorem 2.5 below will allow us to obtain a tight lower bound when k lies "in the middle" of the interval $[0, d]$, by using the comparison between the tropical roots. Unlike the lower bound of Pólya, this theorem gives a bound which is not anymore independent of the coefficients of the polynomial P, although it depends only on a small information, namely a ratio measuring the *seperation* between some tropical roots. Thus, the bound will involve a coefficient $\mathcal{U}(k, \delta)$ depending both on the index k and on the parameter δ (the ratio). This coefficient is defined as follows:

$$\mathcal{U}(k, \delta) := \mathcal{E}(k) \left(\frac{1 + \delta + \sqrt{k^2(1-\delta)^2 + 4\delta}}{1 - \delta} \right) \ , \quad \text{for } k \geqslant 0 \text{ and } 0 \leqslant \delta \leqslant 1 \ , \tag{2.7}$$

where $\mathcal{E}(k)$ is defined by (1.4) for $k \geqslant 1$, and $\mathcal{E}(0) := 1$, and with the convention that $1/0 = +\infty$, so that $\mathcal{U}(k, 1) = +\infty$. It is easy to check that

$$k + \frac{1 + \delta}{1 - \delta} \leqslant \mathcal{U}(k, \delta) < e \left(k + \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} \right) \; . \tag{2.8}$$

We shall consider specially situations in which δ is small. Then, the following asymptotic regime should be kept in mind:

$$\mathcal{U}(k, \delta) \sim \mathcal{E}(k)(k + 1) < e(k + 1), \qquad \delta \to 0 \; .$$

THEOREM 2.5 (Master lower bound). *Let $A_0, \ldots, A_d$, P, $\zeta_1, \ldots, \zeta_{nd}$, $\|\cdot\|$, c, and $\alpha_1, \ldots, \alpha_d$ be as in the first part of Theorem 2.1, and denote $\alpha_0 = 0$ and $\alpha_{d+1} = +\infty$.*
Let $0 \leqslant k^- < k < k^+ \leqslant d + 1$, and denote $\delta_- := \frac{\alpha_{k^-}}{\alpha_k} \leqslant 1$ and $\delta_+ := \frac{\alpha_k}{\alpha_{k^+}} \leqslant 1$.
Then, Inequality (2.1) holds with

$$L_k := \max(L_k^\pm), \quad L_k^+ := \frac{1}{\mathcal{U}(k^+ - k - 1, \delta_+)}, \; L_k^- := \frac{1}{\mathcal{U}(k - k^-, \delta_-)} \; , \tag{2.9}$$

with the convention that $1/\infty = 0$. Moreover, when $\|\cdot\|$ satisfies (A2), the constant L_k of (2.1) can be replaced by the greater constant L_k^*:

$$L_k^* := \max(L_k^{\pm*}), \quad L_k^{+*} := \frac{1}{\sqrt{\mathcal{U}(k^+ - k - 1, \delta_+^2)}}, \; L_k^{-*} := \frac{1}{\sqrt{\mathcal{U}(k - k^-, \delta_-^2)}} \; . \tag{2.10}$$

Theorem 2.5 generalizes Theorem 2.4, and thus the lower bound of Pólya. Indeed, taking $k^- = 0$, $k^+ = d + 1$, and using that $\mathcal{U}(k, 0) = \mathcal{E}(k)(k + 1)$, we get that the constants $L_k^- \leqslant L_k$ and $L_k^{-*} \leqslant L_k^*$ of Theorem 2.5 are exactly the constants L_k of (2.5) and (2.6) of Theorem 2.4 respectively. Moreover, Theorem 2.5 is already new in the scalar case ($n = 1$).

Note that when $\delta_- = 1$ in Theorem 2.5, $\mathcal{U}(k - k^-, \delta_-) = +\infty$, so that $L_k^- = 0$ and $L_k = L_k^+$. Similarly when $\delta_+ = 1$, we get $L_k^+ = 0$. Hence, if $\delta_- = 1 = \delta_+$, we get $L_k = 0$ so that (2.1) does not provide any information, although it is true. However, if for instance $\delta_- = 1$ and $\delta_+ < 1$, we get that $L_k = L_k^+ > 0$, which gives a positive lower bound in (2.1).

Applying Theorem 2.5 to the particular case when $k = k^- + 1 = k^+ - 1$ lead to the following formula for the constants of (2.9) and (2.10):

$$L_k := \max(L_k^\pm), \quad L_k^+ := \frac{1 - \sqrt{\delta_+}}{1 + \sqrt{\delta_+}}, \; L_k^- := \frac{1 - \delta_-}{4(1 + \delta_-)} \; , \tag{2.11}$$

and

$$L_k^* := \max(L_k^{\pm*}), \quad L_k^{+*} := \sqrt{\frac{1 - \delta_+}{1 + \delta_+}}, \; L_k^{-*} := \frac{1}{2} \sqrt{\frac{1 - \delta_-^2}{1 + \delta_-^2}} \; . \tag{2.12}$$

But in this case, one can obtain the following stronger lower bounds.

THEOREM 2.6. *Let us use the notations of Theorem 2.5, and assume that $k = k^- + 1 = k^+ - 1$. Then, the statements of Theorem 2.5 hold with the constants L_k*

and L_k^* *(given in (2.9) and (2.10), or (2.11) and (2.12)) replaced respectively by the greater constants $L_k^\sharp$ and $L_k^{*\sharp}$ given by:*

$$L_k^\sharp := \frac{1 - \delta_- \delta_+}{(1 + \sqrt{\delta_+})^2} \quad , \tag{2.13}$$

and

$$L_k^{*\sharp} := \frac{\sqrt{1 - \delta_-^2 \, \delta_+^2}}{1 + \delta_+} \quad . \tag{2.14}$$

We next indicate how the indices k^+ and k^- should be chosen, for each $1 \leqslant k \leqslant d$, in order to get the best lower bound L_k.

Let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissæ of the vertices of the Newton polygon of $\mathsf{t}p(x)$, as shown in Figure 2.1 (details on the construction of this polygon can be found in Section 3). For $j = 1, \ldots, q$, we have, $\alpha_{k_{j-1}+1} = \cdots = \alpha_{k_j} < \alpha_{k_j+1}$, with the convention $\alpha_{d+1} = +\infty$. We also denote by

$$\delta_j = \frac{\alpha_{k_j}}{\alpha_{k_j+1}} \quad , \tag{2.15}$$

for $j = 0, \ldots, q$, the parameters measuring the *separation* between the tropical roots, in particular $\delta_0 = \delta_q = 0$.
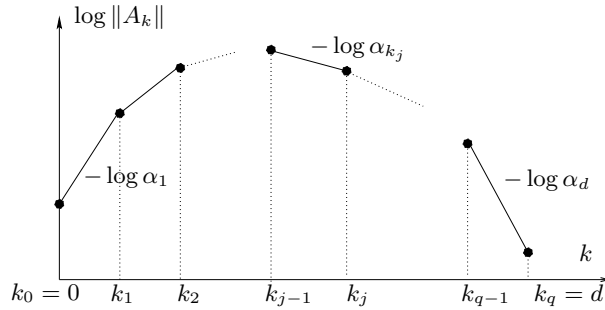

FIG. 2.1. *Newton polygon corresponding to $\mathsf{t}p(x)$.*

PROPOSITION 2.7. *Each of the constants L_k and L_k^* appearing in Theorem 2.5, is maximized by choosing $k^- = k_r$ for some $0 \leqslant r \leqslant q$ such that $k_r < k$ and $k^+ = k_s + 1$ for some $0 \leqslant s \leqslant q$ such that $k_s \geqslant k$.*   This proposition shows that, to apply Theorem 2.5, we may always require k^- and $k^+ - 1$ to be abscissæ of vertices of the Newton polygon. Optimizing the choice of $k^\pm$, we readily arrive at the following corollary.

COROLLARY 2.8.   *Let us use the notations of Theorem 2.5, and let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissæ of the vertices of the Newton polygon of $\mathsf{t}p(x)$, as shown in Figure 2.1. Then, the statements of Theorem 2.5 hold with the constants L_k, $L_k^\pm$, L_k^* and $L_k^{\pm*}$ (given in (2.9) and (2.10)) replaced respectively by the*

following optimal ones:

$$L_k^{opt} := \max(L_k^{\pm,opt}), \tag{2.16a}$$

$$L_k^{+,opt} := \max_{j:\, k_j \geqslant k} \frac{1}{\mathcal{U}(k_j - k, \frac{\alpha_k}{\alpha_{k_j+1}})}, \tag{2.16b}$$

$$L_k^{-,opt} := \max_{j:\, k_{j-1} < k} \frac{1}{\mathcal{U}(k - k_{j-1}, \frac{\alpha_{k_{j-1}}}{\alpha_k})}, \tag{2.16c}$$

and

$$L_k^{*,opt} := \max(L_k^{\pm*,opt}), \tag{2.17a}$$

$$L_k^{+*,opt} := \max_{j:\, k_j \geqslant k} \frac{1}{\sqrt{\mathcal{U}(k_j - k, (\frac{\alpha_k}{\alpha_{k_j+1}})^2)}}, \tag{2.17b}$$

$$L_k^{-*,opt} := \max_{j:\, k_{j-1} < k} \frac{1}{\sqrt{\mathcal{U}(k - k_{j-1}, (\frac{\alpha_{k_{j-1}}}{\alpha_k})^2)}}. \tag{2.17c}$$

However, a simpler choice of $k^{\pm}$ consists in taking the nearest vertices of the Newton polygon in (2.16) and (2.17), which lead to the following corollary.

COROLLARY 2.9. *Let us use the notations of Theorem 2.5, let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissæ of the vertices of the Newton polygon of $\mathbf{t}p(x)$, as shown in Figure 2.1, and let $\delta_0, \ldots, \delta_q$ be defined by (2.15). For $0 \leqslant k \leqslant d$, let us consider the unique $j \in \{1, \ldots, q\}$ such that $k_{j-1} < k \leqslant k_j$, so that $\alpha_k = \alpha_{k_{j-1}+1} = \alpha_{k_j}$. Then, the statements of Theorem 2.5 hold with the constants L_k, $L_k^{\pm}$, L_k^* and $L_k^{\pm*}$ (given in (2.9) and (2.10)) replaced respectively by the following ones:*

$$L_k^{prox} := \max(L_k^{\pm,prox}), \tag{2.18a}$$

$$L_k^{+,prox} := \frac{1}{\mathcal{U}(k_j - k, \delta_j)}, \quad L_k^{-,prox} := \frac{1}{\mathcal{U}(k - k_{j-1}, \delta_{j-1})}, \tag{2.18b}$$

and

$$L_k^{*,prox} := \max(L_k^{\pm*,prox}), \tag{2.19a}$$

$$L_k^{+*,prox} := \frac{1}{\sqrt{\mathcal{U}(k_j - k, \delta_j^2)}}, \quad L_k^{-*,prox} := \frac{1}{\sqrt{\mathcal{U}(k - k_{j-1}, \delta_{j-1}^2)}}. \tag{2.19b}$$

Moreover, in the particular case where $k = k_j$, we have

$$L_k^{prox} \geqslant L_k^{+,prox} = \frac{1 - \sqrt{\delta_j}}{1 + \sqrt{\delta_j}}, \qquad L_k^{*,prox} \geqslant L_k^{+*,prox} = \sqrt{\frac{1 - \delta_j}{1 + \delta_j}}.$$

REMARK 2.10. When all the ratios $\delta_1, \ldots, \delta_q$ are small, the maxima in (2.16) and (2.17) are attained by taking j as in Corollary 2.9, that is $L_k^{\mathrm{opt}} = L_k^{\mathrm{prox}}$ and $L_k^{*,\mathrm{opt}} = L_k^{*,\mathrm{prox}}$ for all $0 \leqslant k \leqslant d$.

REMARK 2.11. Since $\mathcal{U}(k, \delta)$ is increasing in k and δ, the maximizing j in the definition of $L_k^{+*,\mathrm{opt}}$ arises from a compromise between keeping $k_j \geqslant k$ close to k and δ_j small. In particular, when k belongs to an edge of the Newton polygon such that

several consecutive edges have almost the same slope than this edge, the maximizing j may be the one corresponding to the first vertex at which the slope changes significantly, i.e., the first one such that δ_j is small. Similar considerations apply to $L_k^{-*,\text{opt}}$

The previous results provide lower bounds for the ratio between minimal products of modulus of eigenvalues and minimal products of tropical roots. We next state a reverse inequality.

THEOREM 2.12 (Upper bound). *Let $\|\cdot\|$ be any norm on the space of matrices $\mathbb{C}^{n\times n}$. For all $i=1,\ldots,n$, we denote by $A^{(i)}$ the i-th column of A, by η_i the least positive constant such that $\|A^{(i)}\|_2 \leqslant \eta_i\|A\|$ for all $A \in \mathbb{C}^{n\times n}$ where $\|\cdot\|_2$ is the Euclidean norm of $\mathbb{C}^n$, and we set $\eta := \eta_1\cdots\eta_n$. Let $A_0,\ldots,A_d$, P, $\zeta_1,\ldots,\zeta_{nd}$, c, $\mathsf{t}p$, $\alpha_1,\ldots,\alpha_d$, be as in Theorem 2.4, let $k_0 = 0, k_1,\ldots,k_q = d$ be the sequence of all abscissæ of the vertices of the Newton polygon of $\mathsf{t}p(x)$, and define $\delta_0,\ldots,\delta_q$ be as in (2.15). Denote by $C_{n,d,k}$ the number of maps $\phi:\{1,\ldots,n\}\to\{0,\ldots,d\}$ such that $\sum_{j=1}^n \phi(j) = k$, so that $C_{n,d,k} \leqslant (\min(d,k,nd-k)+1)^{n-1}$.*

Then, for every $j=1,\ldots,q$, if

$$c_j := \frac{|\det A_{k_j}|}{\|A_{k_j}\|^n} - (C_{n,d,nk_j} - 1)\eta\delta_j > 0 \ , \tag{2.20}$$

we have

$$\frac{|\zeta_1\ldots\zeta_{nk_j}|}{(\alpha_1\ldots\alpha_{k_j})^n} \leqslant \frac{c}{c_j}\binom{nd}{nk_j} \ .$$

Condition (2.20) in this theorem holds if the matrix A_{k_j} is nonsingular and if the tropical roots α_{k_j} and $\alpha_{k_{j+1}}$ are sufficiently separated, so that $\delta_j = \alpha_{k_j}/\alpha_{k_{j+1}} \ll 1$.

Using the results of Corollary 2.9 and Theorem 2.12, we are able to show that the modulus of a group of eigenvalues corresponding to an edge of the Newton polygon of $\mathsf{t}p$ is bounded from above and below by the corresponding tropical roots. The following immediate corollary gives an example of such bounds.

COROLLARY 2.13. *Under the assumptions and notations of Theorem 2.12, together with (A2), we have, for all $1 \leqslant j \leqslant q$, such that $c_j > 0$ and $c_{j-1} > 0$,*

$$\frac{c_{j-1}}{\binom{nd}{nk_{j-1}}}\left(\frac{1-\delta_j}{1+\delta_j}\right)^{n/2} \leqslant \frac{|\zeta_{nk_{j-1}+1}\ldots\zeta_{nk_j}|}{(\alpha_{k_j})^{n(k_j-k_{j-1})}} \leqslant \frac{\binom{nd}{nk_j}}{c_j}\left(\frac{1+\delta_{j-1}}{1-\delta_{j-1}}\right)^{n/2} \ .$$

Similar results can be written when $\|\cdot\|$ only satisfies (A1), and also for general $1 \leqslant k \leqslant d$. The tightness of the bounds in the above corollary depends on the parameters δ_{j-1}, δ_j and on the condition number of the matrices $A_{k_{j-1}}, A_{k_j}$. Note also that these bounds do not depend on the constant c, except if $j = 1$ (in which case $c_{j-1} = c_0 = c$). Hence, by an argument of continuity, the assumption that $\det A_0 \neq 0$, which is present there (since the notations and assumptions of Theorem 2.12 include the ones of the first part of Theorem 2.1), can be dispensed with, except when $j = 1$.

3. Tropical polynomials and numerical Newton polygons. We recall here basic results on tropical polynomials of one variable. See for instance [BCOQ92, Vir01, IMS07] for more background on tropical polynomials from different perspectives.

Let $\mathbb{R}_{\max}$ denotes the set $\mathbb{R}\cup\{-\infty\}$. A (max-plus) *tropical polynomial* f is a function of a variable $x \in \mathbb{R}_{\max}$ of the form

$$f(x) = \max_{0\leqslant j\leqslant d}(f_j + jx) \ , \tag{3.1}$$

where d is an integer, and $f_0, \ldots, f_d$ are given elements of $\mathbb{R}_{\max}$. We say that f is of *degree* d if $f_d \neq -\infty$. We shall assume that at least one of the coefficients $f_0, \ldots, f_d$ is finite (i.e., that f is not the tropical "zero polynomial"). Then, f is a real valued convex function, piecewise affine, with integer slopes.

Cuninghame-Green and Meijer showed [CGM80] that the analogue of the fundamental theorem of algebra holds in the tropical setting, i.e., $f(x)$ can be written uniquely as

$$f(x) = f_d + \sum_{j=1}^{d} \max(x, \alpha_j) \ ,$$

where $\alpha_1 \leqslant \cdots \leqslant \alpha_d \in \mathbb{R}_{\max}$. The numbers $\alpha_1, \ldots, \alpha_d$ are called the *tropical roots*. The finite tropical roots can be checked to be the points at which the maximum in the expression (3.1) of $f(x)$ is attained at least twice, whereas $-\infty$ arises as a tropical root if $f_0 = -\infty$. The *multiplicity* of a root α is defined as the cardinality of the set $\{j \in \{1, \ldots, j\} \mid \alpha_j = \alpha\}$.

The multiplicity of finite root α can be checked to coincide with the variation of the derivative of the map f at point α, whereas the multiplicity of the root $-\infty$ is given by $\inf\{j \mid f_j \neq -\infty\}$ or by the slope of the map f at $-\infty$. The notion of tropical roots is an elementary special case of the notion of tropical variety which has arisen recently in tropical geometry [IMS07].

The tropical roots can be computed by the following variant of the classical Newton polygon construction. Define the *Newton polygon* $\Delta(f)$ of f to be the upper boundary of the convex hull of the region

$$\{(j, \lambda) \in \mathbb{N} \times \mathbb{R} \mid \lambda \leqslant f_j, \ 0 \leqslant j \leqslant d\} \subset \mathbb{R}^2 \ .$$

This boundary consists of (linear) segments.

The following result was established in [ABG05], it relies on standard Legendre-Fenchel duality argument.

PROPOSITION 3.1 ([ABG05, Proposition 2.10]). *There is a bijection between the set of finite tropical roots of f and the set of segments of the Newton polygon $\Delta(f)$: the tropical root corresponding to a segment is the opposite of its slope, and the multiplicity of this root is the length of this segment (measured by the difference of the abscissæ of its endpoints).* (Actually, min-plus polynomials are considered in [ABG05], but the max-plus case reduces to the min-plus case by an obvious change of variable.)

Since the Graham scan algorithm [Gra72] allows one to compute the convex hull of a finite set of points by making $O(n)$ arithmetical operations and comparisons, provided that the given set of points is already arranged by abscissæ, it follows that the tropical roots, counted with multiplicities, can be computed in linear time (see also [GS09, Proposition 1]). In particular, the maximal tropical root is given by

$$\alpha_d = \max_{0 \leqslant j \leqslant d} \frac{f_j - f_d}{d - j} \ .$$

EXAMPLE 3.2. Consider

$$f(x) = \max(0, 1 + x, 6 + 2x, 4 + 4x, 9 + 8x, 5 + 10x, 1 + 16x) \ .$$

The graph of f and the Newton polygon of f are shown in Figure 3.1. The tropical roots are -3, $-1/2$, and 1, with respective multiplicities 2, 6 and 8.
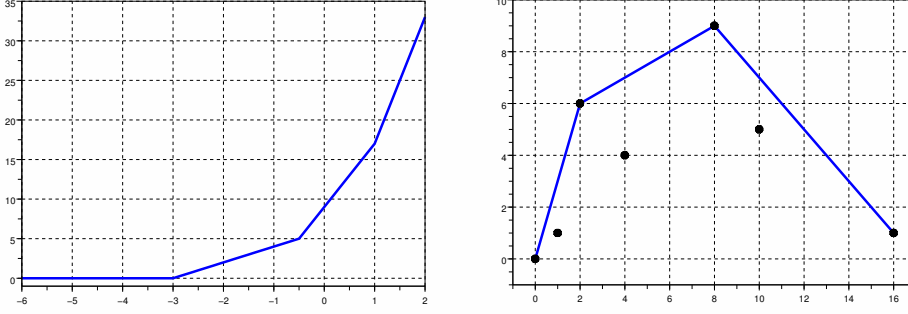
FIG. 3.1. *Graph of the tropical polynomial function f on left, and of its Newton polygon on right.*

The notion of root also applies with trivial changes to the "max-times" model of the tropical structure, in which polynomial functions now have the form

$$\mathsf{t}p(x) = \max_{0 \leqslant j \leqslant d} a_j x^j \ ,$$

where $a_0, \ldots, a_d$ are nonnegative numbers, and the variable x now takes nonnegative values. Then, the *tropical roots* of $\mathsf{t}p(x)$ are, by definition, the exponentials of the tropical roots of its log-exp transformation $f(x) := \log \mathsf{t}p(\exp(x)) = \max_{0 \leqslant j \leqslant d}(\log a_j + jx)$.

In the sequel we shall consider max-times polynomials associated to usual scalar or matrix polynomials. We shall need the following result which follows from the above definitions and properties.

PROPOSITION 3.3. *Let $\mathsf{t}p(x) = \max_{0 \leqslant j \leqslant d} a_j x^j$ be a max-times polynomial of degree d, with $a_j \geqslant 0$, $j = 0, \ldots d$. Assume that $a_0 \neq 0$. Let $0 < \alpha_1 \leqslant \cdots \leqslant \alpha_d$ denote the tropical roots of $\mathsf{t}p$ arranged in non-decreasing order. Then,*

$$a_j \leqslant a_0 \prod_{\ell=1}^{j} \alpha_\ell^{-1}, \quad j = 0, \ldots d \ .$$

Moreover, let k be the abscissa of a vertex of the Newton polygon of p, then $a_k > 0$ and

$$
\begin{aligned}
a_j &\leqslant a_k \textstyle\prod_{\ell=k+1}^{j} \alpha_\ell^{-1}, &&\text{for } j = k, \ldots d \ , \\
a_j &\leqslant a_k \textstyle\prod_{\ell=j+1}^{k} \alpha_\ell, &&\text{for } j = 0, \ldots k \ .
\end{aligned}
$$

Proof. Let $f(x) := \max_{0 \leqslant j \leqslant d}(\log a_j + jx)$. By definition $\log \alpha_j$, $j = 1, \ldots d$, are the tropical roots of f. The upper boundary of the Newton polygon of f coincides with the graph of the concave hull $\hat{a}$ of the map $j \in \{0, \ldots, d\} \mapsto \log a_j \in \mathbb{R}$. By Proposition 3.1, the tropical roots are the opposites of the slopes of $\hat{a}$ and their multiplicities are the lengths of the segments where $\hat{a}$ has this slope. This means that $\log \alpha_j = \hat{a}_{j-1} - \hat{a}_j$, hence $\hat{a}_j = \hat{a}_0 - \sum_{\ell=1}^{j} \log \alpha_\ell$, and using that $\hat{a}$ is above the map $j \mapsto \log a_j$, and that both maps coincide at the boundary point $j = 0$, we get the first inequality of the proposition. If now k is the abscissa of a vertex of the Newton polygon of p, then $(k, \hat{a}_k)$ is an exposed point of the Newton polygon, which implies that $\hat{a}_k = \log a_k$. Since $\hat{a}_j = \hat{a}_k - \sum_{\ell=k+1}^{j} \log \alpha_\ell$, for all $j \geqslant k$, we get the two last inequalities of the proposition. $\square$

4. Tropical bounds for the modulus of the roots of a scalar polynomial.
In this section, we prove the main lower bounds of Section 2 for scalar polynomials,
since the arguments are more transparent in this case. The generalization to the
matrix case, as well as the proof of the upper bound, will be given in the next section.

From Remark 2.2, in the scalar case, all lower bounds reduce to (2.4) with the
constant L_k obtained under Assumption (A2), that is (2.3) in Theorem 2.1, (2.6) in
Theorem 2.4, (2.10) in Theorem 2.5, and (2.14) in Theorem 2.6. The proof of these
lower bounds is based on the next result proved by Landau in [Lan05], building on
an earlier observation of Lindelöf [Lin02]. This result is one key step of the proof of
Pólya's inequality [Ost40a].

LEMMA 4.1 ([Lan05]). *Let $\zeta_1, \ldots, \zeta_d$ be the roots of a polynomial $p(x) = \sum_{j=0}^{d} a_j x^j$,
arranged by non-decreasing modulus, and assume that $a_0 \neq 0$. For all $1 \leqslant k \leqslant d$, we
have*

$$\log |\zeta_1 \ldots \zeta_k| \geqslant - \inf_{r>0} \frac{1}{2} \log \left(\sum_{j=0}^{d} \frac{|a_j|^2}{|a_0|^2} r^{2(j-k)} \right) \quad . \tag{4.1}$$

We include the short proof, as its idea will be used in the extension to the matrix
case given in the next section.

Proof. The formula of Jensen [Jen99] shows that if $\zeta_1, \ldots, \zeta_k$ are the roots of $p(z)$
in the closed disk of $\mathbb{C}$ of radius r, counted with multiplicities, then

$$\log |\zeta_1 \ldots \zeta_k| = k \log r + \log |p(0)| - \frac{1}{2\pi} \int_0^{2\pi} \log |p(re^{i\theta})| d\theta \quad .$$

It follows that, for all $r > 0$, and $k = 1, \ldots, d$:

$$\log |\zeta_1 \ldots \zeta_k| \geqslant k \log r + \log |p(0)| - \frac{1}{2\pi} \int_0^{2\pi} \log |p(re^{i\theta})| d\theta \quad . \tag{4.2}$$

Using the comparison between the geometric and the L^2 mean, together with Parse-
val's identity, we get

$$\frac{1}{2\pi} \int_0^{2\pi} \log |p(re^{i\theta})| d\theta \leqslant \frac{1}{2} \log \left(\frac{1}{2\pi} \int_0^{2\pi} |p(re^{i\theta})|^2 d\theta \right)$$

$$= \frac{1}{2} \log \left(\frac{1}{2\pi} \int_0^{2\pi} | \sum_{j=0}^{d} a_j r^j e^{ij\theta} |^2 d\theta \right)$$

$$= \frac{1}{2} \log \left(\sum_{j=0}^{n} |a_j|^2 r^{2j} \right) \quad .$$

Gathering this inequality with (4.2) yields

$$\log |\zeta_1 \ldots \zeta_k| \geqslant k \log r + \log |a_0| - \frac{1}{2} \log \left(\sum_{j=0}^{n} |a_j|^2 r^{2j} \right) \quad .$$

Since this holds for all $r > 0$, this shows the inequality of the lemma. $\square$

Using the same arguments as in the proof of Pólya's inequality reproduced in [Ost40a],
we obtain the following result which involves now the constants α_k instead of the mod-
ulus $|a_k|$ of the coefficients of p.

COROLLARY 4.2. *Let $\zeta_1, \ldots, \zeta_d$ be the roots of a univariate scalar polynomial p of degree d, $p(x) = \sum_{j=0}^{d} a_j x^j$, arranged by non-decreasing modulus, and assume that $a_0 \neq 0$. Let $\alpha_1, \ldots, \alpha_d$ denote the tropical roots of the associated tropical polynomial $\mathsf{t}p(x) = \max_{0 \leqslant j \leqslant d} |a_j| x^j$, arranged in non-decreasing order. Then, for all $1 \leqslant k \leqslant d$, Inequality (2.4) holds with L_k such that:*

$$(L_k)^{-2} = \inf_{\xi > 0} \left(\sum_{j=0, a_j \neq 0}^{d} \beta_{k,j}^2 \xi^{j-k} \right) , \tag{4.3}$$

with

$$\beta_{k,j} := \begin{cases} \prod_{\ell=j+1}^{k} \left(\frac{\alpha_\ell}{\alpha_k} \right) & \text{if } j < k , \\ \prod_{\ell=k+1}^{j} \left(\frac{\alpha_k}{\alpha_\ell} \right) & \text{if } j > k , \\ 1 & \text{if } j = k , \end{cases} \tag{4.4}$$

which satisfies in particular $\beta_{k,j} \leqslant 1$ for all $k, j = 0, \ldots, d$.

Proof. By applying the change of variable $r = \alpha_k \sqrt{\xi}$ in the inequality of Lemma 4.1, we get

$$\log |\zeta_1 \ldots \zeta_k| \geqslant \sup_{\xi > 0} \left(-\frac{1}{2} \log \left(\sum_{j=0}^{d} \frac{|a_j|^2}{|a_0|^2} (\alpha_k^2 \xi)^{j-k} \right) \right)$$

$$\geqslant \log(\alpha_1 \cdots \alpha_k) + \sup_{\xi > 0} \left(-\frac{1}{2} \log \left(\prod_{\ell=1}^{k} \alpha_\ell^2 \left(\sum_{j=0}^{d} \frac{|a_j|^2}{|a_0|^2} (\alpha_k^2 \xi)^{j-k} \right) \right) \right) .$$

Applying Proposition 3.3 to the max-times polynomial $\mathsf{t}p$, we get $|a_j| \leqslant |a_0| \prod_{\ell=1}^{j} \alpha_\ell^{-1}$ for all $j = 0, \ldots, d$, which with the above inequality yields (2.4) with

$$(L_k)^{-2} := \inf_{\xi > 0} \left(\sum_{j=0, a_j \neq 0}^{d} \left(\frac{(\prod_{\ell=1}^{k} \alpha_\ell)(\prod_{\ell=1}^{j} \alpha_\ell^{-1})}{\alpha_k^{k-j}} \right)^2 \xi^{j-k} \right)$$

which can be written in the form (4.3) with $\beta_{k,j}$ as in (4.4). Moreover, since α_j is nondecreasing with respect to j, we get that all the $\beta_{k,j}$ are less than or equal to 1. □

Pólya's inequality is then an immediate consequence of Corollary 4.2. Indeed, using the property that all the $\beta_{k,j}$ are less than or equal to 1, we obtain:

$$(L_k)^{-2} \leqslant \sum_{j=0}^{d} \xi^{j-k} \leqslant \sum_{j=0}^{\infty} \xi^{j-k} = \frac{1}{\xi^k (1-\xi)} ,$$

for all $\xi > 0$. The minimum of the right hand side of the previous inequality for $0 < \xi < 1$, is attained for $\xi = k/(k+1)$, from which we deduce (1.3), which is also the scalar version of Theorem 2.4.

We can now deduce similarly, from Corollary 4.2, the scalar versions of the lower bounds of Theorems 2.1, 2.5, and 2.6. Since in the scalar case, we are reduced to show (2.4) for some constants L_k, and that this inequality is precisely the statement of Corollary 4.2, we only need to show that these constants L_k are lower bounds of the constant L_k of (4.3).

Proof of the scalar version of Theorem 2.1. Using the property that $\beta_{k,j} \leqslant 1$ for all k, j, we obtain that the constant L_k of (4.3) satisfies $(L_k)^{-2} \leqslant \sum_{j=0, a_j \neq 0}^{d} \xi^{j-k}$ for all $\xi > 0$. When $\xi = 1$, the right hand side of this inequality is equal to the number of non-zero coefficients of p, which shows that the constant L_k of (4.3) is lower bounded by the constant L_k of (2.3), which implies (2.4) with this lower bound L_k. $\square$

Proof of the scalar version of Theorem 2.5. Assume that $0 \leqslant k^- < k < k^+ \leqslant d+1$. Denote $\delta_- = \alpha_{k^-}/\alpha_k$ and $\delta_+ = \alpha_k/\alpha_{k^+}$. Then, for all $1 \leqslant \ell \leqslant k$, $\alpha_\ell \leqslant \alpha_k$, and for all $1 \leqslant \ell \leqslant k^-$, $\alpha_\ell \leqslant \alpha_{k^-} = \alpha_k \delta_-$. This implies that $\beta_{k,j} \leqslant \delta_-^{k^- -j}$, for all $j \leqslant k^-$. Similarly, for all $\ell \geqslant k$, $\alpha_\ell \geqslant \alpha_k$, and for all $\ell \geqslant k^+$, $\alpha_\ell \geqslant \alpha_{k^+} = \alpha_k/\delta_+$, hence, for all $j \geqslant k^+$, $\beta_{k,j} \leqslant \delta_+^{j-k^+ +1}$. Since we also have $\beta_{k,j} \leqslant 1$ for all j, k, we obtain that the constant L_k of (4.3) satisfies, for all $\delta_-^2 < \xi < \delta_+^{-2}$, $\xi \neq 1$,

$$(L_k)^{-2} \leqslant \left(\sum_{j=0}^{k^-} \delta_-^{2(k^- -j)} \xi^{j-k} \right) + \left(\sum_{j=k^-+1}^{k^+-1} \xi^{j-k} \right) + \left(\sum_{j=k^+}^{d} \delta_+^{2(j-k^+ +1)} \xi^{j-k} \right)$$

$$\leqslant \xi^{k^- -k+1} \frac{1}{\xi - \delta_-^2} + \frac{\xi^{k^- -k+1} - \xi^{k^+ -k}}{1 - \xi} + \xi^{k^+ -k-1} \frac{\delta_+^2 \xi}{1 - \delta_+^2 \xi} .$$

Since $\frac{\delta_+^2 \xi}{1-\delta_+^2 \xi} \leqslant \frac{\xi}{1-\delta_+^2 \xi}$ the later inequality can be written as

$$(L_k)^{-2} \leqslant g(\xi), \quad \forall \delta_-^2 < \xi < \delta_+^{-2}, \ \xi \neq 1, \tag{4.5}$$

with

$$g(\xi) := g_-(\xi) + g_+(\xi) \ ,$$

$$g_-(\xi) := \xi^{k^- -k+1} \left(\frac{1}{\xi - \delta_-^2} + \frac{1}{1 - \xi} \right) \ ,$$

$$g_+(\xi) := \xi^{k^+ -k-1} \left(\frac{1}{\xi - 1} + \frac{1}{1 - \delta_+^2 \xi} \right) \ .$$

Note that (4.5) also holds when $k^- = 0$ or $k^+ = d + 1$, since then $\delta_- = 0$ or $\delta_+ = 0$ respectively. When $\delta_- = 1$ and $\delta_+ < 1$, the conditions on ξ in (4.5) are equivalent to $1 < \xi < \delta_+^{-2}$, whereas when $\delta_- = \delta_+ = 1$, these conditions are never satisfied, but in this case the constant L_k of (2.10) is equal to 0 so there is nothing to prove.

The functions g_- and g_+ satisfy $g_-(\xi) = g_{k-k^-, \delta_-^2}(\xi^{-1})$ and $g_+(\xi) = g_{k^+ -k-1, \delta_+^2}(\xi)$ where $g_{k,\delta}$ is defined for $\xi \neq 1$ and δ^{-1} by:

$$g_{k,\delta}(\xi) := \xi^k \left(\frac{1}{\xi - 1} + \frac{1}{1 - \delta \xi} \right) = \xi^{k+1} \frac{1 - \delta}{(\xi - 1)(1 - \delta \xi)} \ .$$

We have $g_{k,\delta}(\xi) \leqslant 0$ for $\xi < 1$, hence $g_-(\xi) \leqslant 0$ for all $\xi > 1$, and $g_+(\xi) \leqslant 0$ for all $\xi < 1$. When $\delta < 1$, the minimum of $g_{k,\delta}$ on $(1, \delta^{-1})$ is attained at

$$\xi_{k,\delta} := \begin{cases} \frac{k(\delta+1) - \sqrt{k^2(\delta+1)^2 - 4\delta(k^2-1)}}{2(k-1)\delta} & \text{when } k \neq 1, \ \delta \neq 0 \ , \\ \frac{2}{1+\delta} & \text{when } k = 1 \ , \\ \frac{k+1}{k} & \text{when } \delta = 0 \ . \end{cases}$$

The last formula gives $\xi_{k,\delta} = \infty$ when $\delta = 0$ and $k = 0$, which is the point of infimum of $g_{k,\delta}$ for $1 < \xi < \infty = \delta^{-1}$, since $g_{k,\delta}$ is decreasing. It also gives $\xi_{k,1} = 1$ for all $k \geqslant 0$.

Hence extending $g_{k,\delta}$, g_- and g_+ by $+\infty$ at point 1, we get that the infimum of $g_{k,\delta}$ on $(1, \delta^{-1})$ equals $g_{k,\delta}(\xi_{k,\delta})$, and denoting $\xi_- = (\xi_{k-k^-,\delta_-^2})^{-1}$ and $\xi_+ = \xi_{k^+-k-1,\delta_+^2}$, we obtain

$$(L_k)^{-2} \leqslant \min(g(\xi_-), g(\xi_+)) \leqslant \min(g_-(\xi_-), g_+(\xi_+)) \ . \tag{4.6}$$

In order to simplify this bound, we need to find good estimates of $\xi_{k,\delta}$.

When $k \neq 1$ and $\delta \neq 0$, we have:

$$\xi_{k,\delta} = \frac{k(\delta+1) - \sqrt{k^2(1-\delta)^2 + 4\delta}}{2(k-1)\delta}$$

$$= \frac{2(k+1)}{k(\delta+1) + \sqrt{k^2(1-\delta)^2 + 4\delta}} \ .$$

Using the property that $\delta \geqslant 0$ in the last formula, we get that $\xi_{k,\delta} \leqslant \frac{k+1}{k} = \xi_{k,0}$. Moreover, this inequality also holds for $k = 1$. In particular $(\xi_{k,\delta})^k \leqslant \mathcal{E}(k)$, for all $k \geqslant 0$ and $\delta \geqslant 0$ (taking the convention that $\xi^0 = 1$ for all $\xi \in (1, \infty]$). We also have, for $k \geqslant 0$ and $0 \leqslant \delta < 1$,

$$\frac{1}{\xi_{k,\delta} - 1} + \frac{1}{1 - \delta\xi_{k,\delta}} = \frac{1 + \delta + \sqrt{k^2(1-\delta)^2 + 4\delta}}{1 - \delta} \leqslant k + \frac{1 + \sqrt{\delta}}{1 - \sqrt{\delta}} \ .$$

This yields

$$g_{k,\delta}(\xi_{k,\delta}) \leqslant \mathcal{U}(k, \delta) \quad \forall k \geqslant 0, \ 0 \leqslant \delta \leqslant 1,$$

with $\mathcal{U}$ as in (2.7). Indeed, this inequality holds for $\delta < 1$ by the above arguments. It also holds for $\delta = 1$ since $\mathcal{U}(k, 1) = +\infty$. This implies in particular that $g_+(\xi_+) \leqslant \mathcal{U}(k^+ - k - 1, \delta_+^2)$ and $g_-(\xi_-) \leqslant \mathcal{U}(k - k^-, \delta_-^2)$. Combining these two inequalities with (4.6), we obtain that the constant L_k of (4.3) is lower bounded by the constant L_k^* of (2.10), which implies (2.4) with this lower bound L_k^* instead of L_k. $\square$

REMARK 4.3. In the previous proof, obtaining the minimum of g instead of g_- and g_+ would have led to a better lower bound for L_k than the one of Theorem 2.5. However, such a bound is more difficult to estimate for general values of $k^+ - k^-$. We can use for instance the first inequality in (4.6), which will give better estimates in some particular cases. For instance, when $k = 0$, we get that $g_{k,\delta}(\xi_{k,\delta}) = \mathcal{U}(k, \delta) = (1 + \sqrt{\delta})/(1 - \sqrt{\delta})$. In particular, when $k = k^+ - 1$, the bound $(L_k)^{-2} \leqslant g_+(\xi_+)$ gives the lower bound L_k^+ of (2.12), with $\delta = \delta_+$. However (4.6) gives the slightly better bound:

$$(L_k)^{-2} \leqslant g(\xi_+) = \frac{1 + \delta_+}{1 - \delta_+} - \frac{1 - \delta_-^2}{(1 - \delta_+\delta_-^2)(1 - \delta_+)}\delta_+^{k^+-k^-} \ .$$

Proof of the scalar version of Theorem 2.6. Let us use the notation of the previous proof. When $k = k^+ - 1 = k^- + 1$, Inequality (4.5) is true for all $\xi \in (\delta_-^2, \delta_+^{-2})$ since g can be well defined at 1 (by continuity for instance). When $\delta_-\delta_+ < 1$, the interval $(\delta_-^2, \delta_+^{-2})$ is nonempty, the map g is convex there and its minimum is achieved at the point $\xi = (1 + \delta_-^2\delta_+)/(\delta_+(1 + \delta_+))$. Then (4.5) yields $(L_k)^{-2} \leqslant (1 + \delta_+)^2/(1 - \delta_-^2\delta_+^2)$, which implies that the constant L_k of (4.3) is lower bounded by the constant $L_k^{*\sharp}$ of (2.14), which implies (2.4) with this lower bound $L_k^{*\sharp}$ instead of L_k. Moreover,

since the minimum of g is less or equal to the right hand side of (4.6), the proof of the scalar version of Theorem 2.5 implies that $L_k^{*\sharp} \leqslant L_k^*$. $\square$

EXAMPLE 4.4. Consider the following scalar polynomial

$$p(x) = 1 - \exp(1)x - \exp(6)x^2 + \exp(4)x^4 + \exp(9)x^8 + \exp(5)x^{10} + \exp(1)x^{16} \ . \ (4.7)$$

The log-exp transformation of its tropical polynomial $\mathsf{t}\,p$ is the tropical polynomial f of Example 3.2. The graph of f and the associated Newton polygon were shown in Figure 3.1. Hence, the tropical roots of $\mathsf{t}\,p$ are the exponentials of -3, $-1/2$, and 1, with multiplicities 2, 6 and 8, respectively. In Table 4.1, we are comparing the value of the ratio $\frac{|\zeta_1\ldots\zeta_k|}{\alpha_1\ldots\alpha_k}$, with the lower bound of Pólya given in (1.3), together with the lower bounds obtained in Theorem 2.1 and Corollary 2.9. There, for each $k = 1, \ldots, d = 16$, the column RATIO gives $\frac{|\zeta_1\ldots\zeta_k|}{\alpha_1\ldots\alpha_k}$, the column PÓLYA gives the lower bound of Pólya, which coincides with the constant L_k of (2.6), the column FEWNOM gives the lower bound L_k of (2.3), and the column SEPAR gives the lower bound $L_k^{*,\mathrm{prox}}$ of (2.19), based on the separation between the tropical roots.

k	RATIO	FEWNOM	PÓLYA	SEPAR
1	0.93475	0.37796	0.5	0.5
2*	1.00034	"	0.3849	0.92102
3	0.98782	"	0.32476	0.49664
4	0.97926	"	0.28622	0.38360
5	0.98348	"	0.25880	0.32403
6	0.98771	"	0.23802	0.37508
7	0.99383	"	0.22156	0.47570
8*	1.00000	"	0.20810	0.79696
9	0.98811	"	0.19683	0.47570
10	0.97636	"	0.18721	0.37508
11	0.96476	"	0.17888	0.31917
12	0.95329	"	0.17158	0.28622
13	0.96476	"	0.16510	0.32476
14	0.97636	"	0.15930	0.3849
15	0.98811	"	0.15407	0.5
16*	1	"	0.14933	1

TABLE 4.1

*Comparison of the lower bounds for the scalar polynomial p of (4.7). The abscissæ of the vertices of the Newton polygon of $\mathsf{t}\,p$ are indicated by the symbol *.*

EXAMPLE 4.5. Consider now the following scalar polynomial

$$\begin{aligned} p(x) = &1 - \exp(3)x - \exp(6)x^2 - \exp(7)x^4 - \exp(8)x^6 + \exp(9)x^8 \\ &+ \exp(7)x^{10} - \exp(4)x^{13} - \exp(3)x^{14} + \exp(1)x^{16} \ . \end{aligned} \qquad (4.8)$$

The log-exp transformation of its tropical polynomial $\mathsf{t}\,p$ is again the tropical polynomial f of Example 3.2, but now there are some points of the graph of the map $k \mapsto \log|p_k|$, where p_k is the kth coefficient of p, that are on the edges of its Newton polygon, that is its concave hull. Table 4.2 is obtained on the same principle as Table 4.1. In Figure 4.5, we plot in the same graph the values of the ratios and lower bounds for the scalar polynomials p of (4.7) and (4.8). This indicates that the lower bound L_k of Corollary 2.9, based on the separation between tropical roots, may become tighter when the polynomial has non zero coefficients between the vertices of the Newton polygon.

5. Proofs of the bounds for the modulus of the eigenvalues of a matrix polynomial. In this section we provide the proofs of the main results which were stated in Section 2 and check the properties given at the beginning of Section 2 on

k	RATIO	FEWNOM	PÓLYA	SEPAR
1	0.61759	0.31623	0.5	0.5
2*	0.98684	"	0.3849	0.92102
3	0.84948	"	0.32476	0.49664
4	0.73124	"	0.28622	0.38360
5	0.63280	"	0.25880	0.32403
6	0.54760	"	0.23802	0.37508
7	0.72118	"	0.22156	0.47570
8*	0.95684	"	0.20810	0.79696
9	0.76959	"	0.19683	0.47570
10	0.61899	"	0.18721	0.37508
11	0.57724	"	0.17888	0.31917
12	0.53832	"	0.17158	0.28622
13	0.62117	"	0.16510	0.32476
14	0.71677	"	0.15930	0.3849
15	0.84662	"	0.15407	0.5
16*	1	"	0.14933	1

TABLE 4.2

Comparison of the lower bounds for the scalar polynomial p of (4.8). *The abscissæ of the vertices of the Newton polygon of* tp *are indicated by the symbol* *.
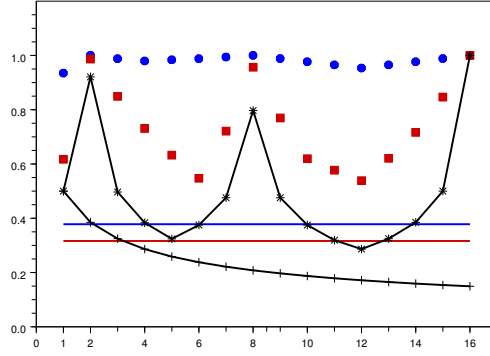

FIG. 4.1. *Plot of* RATIO *as a function of k for the scalar polynomials of* (4.7) *(blue circles) and* (4.8) *(red squares), together with the lower bound* FEWNOM *(blue and red lines resp.),* PÓLYA *(plus signs) and* SEPAR *(stars signs).*

norms on the space of matrices. For all $p \in [1, \infty]$, and $n \geqslant 1$, we shall denote by $\|\cdot\|_p$ the ℓ^p norm of $\mathbb{C}^n$: $\|v\|_p := (\sum_{i=1}^n |v_i|^p)^{1/p}$ for $p < \infty$ and $\|v\|_\infty := \max_{i=1,\ldots,n} |v_i|$. In particular as in Theorem 2.12, $\|\cdot\|_2$ is the Euclidean norm. The norm on the space of matrices $\mathbb{C}^{n \times n}$ induced by the norm $\|\cdot\|_p$ on $\mathbb{C}^n$ (the same norm is used for the domain and the range of A) will also be denoted by $\|\cdot\|_p$. In particular the norm $\|\cdot\|_2$ on $\mathbb{C}^{n \times n}$ is the spectral norm. Moreover, for all $p \in [1, \infty)$, and $n \geqslant 1$, we shall denote by $\|\cdot\|_{*p}$ the following normalized Schatten p-norm on the space of matrices $\mathbb{C}^{n \times n}$: $\|A\|_{*p} := (\frac{1}{n} \sum_{i=1}^n \sigma_i^p)^{1/p}$, where $\sigma_1, \ldots, \sigma_n$ are the singular values of A (the eigenvalues of $\sqrt{A^*A}$, where A^* denotes the conjugate transpose of A). Then, the normalized Schatten 2-norm coincides with the normalized Frobenius norm. Recall that the (unnormalized) Schatten 1-norm is also called the trace norm or the Ky Fan n-norm. We shall also denote by $\|A\|_{*\infty} := \max_{i=1,\ldots,n} \sigma_i$ the Schatten ∞-norm, which coincides with the spectral norm.

PROPERTY 5.1. *The normalized Frobenius norm satisfies Assumption* (A1). *Therefore, Assumption* (A2) *implies Assumption* (A1).

Proof. From Hadamard's inequality, we get that $|\det A| \leqslant \|A^{(1)}\|_2 \cdots \|A^{(n)}\|_2$, for all $A \in \mathbb{C}^{n\times n}$, where $A^{(j)}$ denotes the j-th column of A. Then, since the geometric mean is less than or equal to the arithmetic mean (or simply by the concavity of the logarithm), we get that $|\det A|^{2/n} \leqslant \frac{1}{n}(\|A^{(1)}\|_2^2 + \cdots + \|A^{(n)}\|_2^2) = \|A\|_{\mathrm{nF}}^2$, which implies that the normalized Frobenius norm satisfies Assumption (A1). Since for all $Q, Q' \in \mathbb{C}^{n\times n}$ such that $\det Q = \det Q' = 1$, we get $|\det A| = |\det(QAQ')| \leqslant \|QAQ'\|_{\mathrm{nF}}^n$ for all $A \in \mathbb{C}^{n\times n}$, Assumption (A2) implies Assumption (A1). □

PROPERTY 5.2. *For all $p \in [1,\infty]$, the norm $\|\cdot\|_p$ on $\mathbb{C}^{n\times n}$ satisfies* (A2).

Proof. Indeed, let us denote by $\|A\|_{p,q}$ the norm induced by the norms $\|\cdot\|_p$ and $\|\cdot\|_q$ on the range and domain $\mathbb{C}^n$ of A respectively, which means that $\|A\|_{p,q} = \max\{\|Av\|_q/\|v\|_p, v \in \mathbb{C}^n, v \neq 0\}$. Since (for all $v \in \mathbb{C}^n$) $p \in [1,\infty] \mapsto \|v\|_p \in \mathbb{R}^+$ is a nonincreasing map, we get that $\|A\|_{p,q} \leqslant \|A\|_{p',q'}$ when $p \leqslant p'$ and $q \geqslant q'$. In particular $\|A\|_p = \|A\|_{p,p} \geqslant \|A\|_{2,\infty}$ when $2 \leqslant p$, and $\|A\|_p = \|A\|_{p,p} \geqslant \|A\|_{1,2}$ when $p \leqslant 2$. In addition, it is easy to show that $\|A\|_{2,\infty} = \max\{\|(A^*)^{(i)}\|_2, i = 1,\ldots,n\}$. Since the last expression is greater or equal to $\|A\|_{\mathrm{nF}}$, we deduce that $\|A\|_p \geqslant \|A\|_{2,\infty} \geqslant \|A\|_{\mathrm{nF}}$ for all $A \in \mathbb{C}^{n\times n}$ and $2 \leqslant p \leqslant \infty$. Similarly, $\|A\|_{1,2} = \max\{\|A^{(i)}\|_2, i = 1,\ldots n\} \geqslant \|A\|_{\mathrm{nF}}$, hence $\|A\|_p \geqslant \|A\|_{1,2} \geqslant \|A\|_{\mathrm{nF}}$ for all $A \in \mathbb{C}^{n\times n}$ and $1 \leqslant p \leqslant 2$. This shows that $\|\cdot\|_p$ satisfies (A2) with $Q = Q' = I$ the identity matrix. □

This result implies that all norms induced by the norm $\|v\|_d = \|Qv\|_p$ in the domain of A and the norm $\|v\|_r = \|Q'v\|_p$ in the range of A with the same p, but possibly different matrices Q, Q' such that $\det Q = \det Q' = 1$, satisfy (A2).

PROPERTY 5.3. *If $\|\cdot\|$ is the norm on $\mathbb{C}^{n\times n}$ induced by a norm on $\mathbb{C}^n$ (the same for the domain and the range of matrices), then it satisfies* (A1).

Proof. Since $\det A$ is the product of the eigenvalues of A counted with multiplicities, we get that $|\det A| \leqslant \rho(A)^n$, where $\rho(A)$ is the spectral radius of A. Since $\rho(A) \leqslant \|A\|$ for all $A \in \mathbb{C}^{n\times n}$ and all induced norms $\|\cdot\|$ on the space of matrices, we get the result. □

PROPERTY 5.4. *The normalized Schatten p-norm $\|\cdot\|_{*p}$ on $\mathbb{C}^{n\times n}$ satisfies* (A1), *for all $p \in [1,\infty]$. It satisfies* (A2) *if and only if $p \geqslant 2$. Moreover, for $p \in [1,2)$, the least constant η such that $\eta\|\cdot\|_{*p}$ satisfies* (A2) *is given by $\eta = n^{1/p-1/2} > 1$.*

Proof. Since $|\det A|$ is the product of the singular values of A counted with multiplicities, we get that $|\det A|^p = \sigma_1^p \cdots \sigma_n^p$, and using that the geometric mean is less than or equal to the arithmetic mean, we obtain that $|\det A|^{p/n} \leqslant \frac{1}{n}(\sigma_1^p + \cdots + \sigma_n^p) = \|A\|_{*p}^p$, which implies that the normalized Schatten p-norm satisfies Assumption (A1).

We have that $p \mapsto \|A\|_{*p}$ is a nondecreasing map. Hence, when $p \in [2,\infty]$, $\|A\|_{*p} \geqslant \|A\|_{*2} = \|A\|_{\mathrm{nF}}$, for all $A \in \mathbb{C}^{n\times n}$, which implies that $\|\cdot\|_{*p}$ satisfies (A2) when $p \geqslant 2$. Let $\eta > 0$ be the least constant such that $\eta\|\cdot\|_{*p}$ satisfies (A2) and let us show that $\eta = n^{1/p-1/2}$ when $p < 2$. This will implies in particular that $\eta > 1$ hence $\|\cdot\|_{*p}$ does not satisfy (A2) for $p < 2$, which will finishes the proof of the equivalence "$\|\cdot\|_{*p}$ satisfies (A2) if and only if $p \geqslant 2$". Since $n^{1/p}\|\cdot\|_{*p}$ is the unnormalized Schatten norm, which is nonincreasing with respect to p, we get that, for all $p \in [1,2)$, and $A \in \mathbb{C}^{n\times n}$, $n^{1/2}\|A\|_{\mathrm{nF}} \leqslant n^{1/p}\|A\|_{*p}$, which implies that $\eta \leqslant n^{1/p-1/2}$. Let us fix p, $Q, Q' \in \mathbb{C}^{n\times n}$ such that $\det Q = \det Q' = 1$ and $\|QAQ'\|_{\mathrm{nF}} \leqslant \eta\|A\|_{*p}$, for all $A \in \mathbb{C}^{n\times n}$. For all $i = 1,\ldots,n$, let us consider the matrix A whose entries are all zero but the entry ii equal to 1. Then, the singular values of A are all zero except one which is equal to 1, so that $\|A\|_{*p} = (1/n)^{1/p}$. Moreover $QAQ' = Q^{(i)}Q'_{(i)}$, where $Q'_{(i)}$ denotes the i-th raw of Q'. Hence $\|QAQ'\|_{\mathrm{nF}} = (1/n)^{1/2}\|Q^{(i)}\|_2\|(Q'^*)^{(i)}\|_2$. From $\|QAQ'\|_{\mathrm{nF}} \leqslant \eta\|A\|_{*p}$,

we deduce that $\|Q^{(i)}\|_2\|(Q'^*)^{(i)}\|_2 \leqslant \eta(1/n)^{1/p-1/2}$. Since $\det Q = \det Q' = 1$, we get using Hadamard's inequality, $1 \leqslant \|Q^{(1)}\|_2\|(Q'^*)^{(1)}\|_2 \cdots \|Q^{(n)}\|_2\|(Q'^*)^{(n)}\|_2 \leqslant (\eta(1/n)^{1/p-1/2})^n$, which shows that $\eta \geqslant n^{1/p-1/2}$. We deduce that $\eta = n^{1/p-1/2} > 1$ for $p > 2$, which completes the proof. $\square$

For the proof of lower bound results stated in Section 2, the approach is similar to the scalar case, except that we change the application of Jensen formula to obtain the following matrix version of Lemma 4.1.

LEMMA 5.5. *Let $\zeta_1,\ldots,\zeta_{nd}$ denote the eigenvalues of the matrix polynomial P with degree d:*

$$P(\lambda) = A_0 + A_1\lambda + \cdots + A_d\lambda^d \quad A_j \in \mathbb{C}^{n\times n} \quad 0 \leqslant j \leqslant d \ ,$$

arranged by non-decreasing modulus. Assume that $\|\cdot\|$ is any norm on the space of matrices which satisfies (A1), *that $\det A_0 \neq 0$ and let $c = \frac{|\det A_0|}{\|A_0\|^n}$. Then, for $1 \leqslant k \leqslant d$, we have*

$$\log|\zeta_1\ldots\zeta_{nk}| \geqslant \log c - n \inf_{r>0} \log\left(\sum_{j=0}^{d}\frac{\|A_j\|}{\|A_0\|}r^{j-k}\right) \ . \tag{5.1}$$

When $\|\cdot\|$ satisfies (A2), *the previous bound can be improved as follows*

$$\log|\zeta_1\ldots\zeta_{nk}| \geqslant \log c - n \inf_{r>0}\frac{1}{2}\log\left(\sum_{j=0}^{d}\frac{\|A_j\|^2}{\|A_0\|^2}r^{2(j-k)}\right) \ . \tag{5.2}$$

Proof. From Inequality (4.2) applied to $\tilde{p}(\lambda) = \det P(\lambda)$, we get, for all $r > 0$,

$$\log|\zeta_1\ldots\zeta_{nk}| \geqslant nk\log r + \log|\det A_0| - \frac{1}{2\pi}\int_0^{2\pi}\log|\det P(re^{i\theta})|d\theta \ . \tag{5.3}$$

Since $\|\cdot\|$ satisfies (A1), we have

$$|\det P(re^{i\theta})| \leqslant \|P(re^{i\theta})\|^n = \|\sum_{j=0}^{d}A_j(re^{i\theta})^j\|^n \leqslant (\sum_{j=0}^{d}\|A_j\|r^j)^n \ , \tag{5.4}$$

for all $\theta \in [0,2\pi]$. Gathering (5.4) with (5.3), we obtain (5.1).

Assume now that $\|\cdot\|$ satisfies (A2) with some matrices Q, Q' such that $\det Q = \det Q' = 1$, and let us prove (5.2). Since from Property 5.1, the normalized Frobenius norm satisfies (A1), we get that $|\det P(re^{i\theta})| = |\det(QP(re^{i\theta})Q')| \leqslant \|QP(re^{i\theta})Q'\|_{\mathrm{nF}}^n$. Now using the comparison between geometric and L^2 means, we deduce

$$\frac{1}{2\pi}\int_0^{2\pi}\log|\det P(re^{i\theta})|d\theta \leqslant n\frac{1}{2\pi}\int_0^{2\pi}\log\|QP(re^{i\theta})Q'\|_{\mathrm{nF}}d\theta$$

$$\leqslant \frac{n}{2}\log\left(\frac{1}{2\pi}\int_0^{2\pi}\|QP(re^{i\theta})Q'\|_{\mathrm{nF}}^2 d\theta\right) \ . \tag{5.5}$$

From the formula of the normalized Frobenius norm, we get by applying Parseval's

identity to each coordinate $(QP(re^{i\theta})Q')_{\ell m}$

$$\frac{1}{2\pi}\int_0^{2\pi}\|QP(re^{i\theta})Q'\|_{\mathrm{nF}}^2 d\theta = \frac{1}{n}\sum_{\ell,m=1}^n\left(\frac{1}{2\pi}\int_0^{2\pi}|(QP(re^{i\theta})Q')_{\ell,m}|^2 d\theta\right)$$

$$= \frac{1}{n}\sum_{\ell,m=1}^n\left(\sum_{j=0}^d(|(QA_jQ')_{\ell,m}|r^j)^2\right)$$

$$= \sum_{j=0}^d(\|QA_jQ'\|_{\mathrm{nF}}^2 r^{2j})$$

$$\leqslant \sum_{j=0}^d(\|A_j\|^2 r^{2j}) \ .$$

Gathering this inequality with Inequalities (5.5) and (5.3), we obtain (5.2). □

Using the same arguments as for Corollary 4.2, we obtain the following result.

COROLLARY 5.6. *Let* $\zeta_1,\ldots,\zeta_{nd}$ *denote the eigenvalues of the matrix polynomial* P *with degree* d:

$$P(\lambda) = A_0 + A_1\lambda + \cdots + A_d\lambda^d \quad A_j \in \mathbb{C}^{n\times n} \quad 0\leqslant j\leqslant d \ ,$$

arranged by non-decreasing modulus. Assume that $\|\cdot\|$ *is any norm on the space of matrices which satisfies* (A1)*, that* $\det A_0 \neq 0$ *and let* $c = \frac{|\det A_0|}{\|A_0\|^n}$*. Let* $\alpha_1,\ldots,\alpha_d$ *be the tropical roots of the following tropical polynomial:*

$$\mathsf{t}\,p(x) = \max_{0\leqslant j\leqslant d}(\|A_j\|x^j) \ ,$$

arranged in non-decreasing order. Then, for all $1\leqslant k\leqslant d$*, Inequality* (2.1) *holds with* L_k *such that:*

$$(L_k)^{-1} = \inf_{\xi>0}\left(\sum_{j=0,A_j\neq 0}^d\beta_{k,j}\xi^{j-k}\right) \ , \tag{5.6}$$

with $\beta_{k,j}$ *as in* (4.4)*.*

Moreover, when $\|\cdot\|$ *satisfies* (A2)*, the constant* L_k *in* (2.1) *can be replaced by the greater one:*

$$(L_k)^{-2} = \inf_{\xi>0}\left(\sum_{j=0,A_j\neq 0}^d\beta_{k,j}^2\xi^{j-k}\right) \ . \tag{5.7}$$

Proof of Theorems 2.1, 2.4, 2.5, and 2.6. In Corollary 5.6, the formula (5.7) of L_k coincides with (4.3), except that the condition $a_j \neq 0$ is replaced by $A_j \neq 0$. Hence, the lower bounds of the constant L_k of (4.3) obtained in the proofs of the scalar versions of Theorems 2.1, 2.5, and 2.6 give similar lower bounds for the constant L_k of (5.7), which combined with the assertion that Inequality (2.1) holds with this constant, now provide the lower bounds of Theorems 2.1, 2.5, and 2.6 respectively, in the case where $\|\cdot\|$ satisfies (A2), that is Inequality (2.1) with L_k replaced by

the constant L_k of (2.3), the constant L_k^* of (2.10), and the constant $L_K^{*\sharp}$ of (2.14), respectively.

In the case of a norm satisfying only (A1), the formula (5.7) of L_k is replaced by (5.6), which means that everything behave as if L_k were replaced by its square and the numbers α_k were replaced by their square roots in (5.7). Hence the assertions of Theorems 2.1, 2.5, and 2.6 in the case of Assumption (A2), are still true under Assumption (A1), up to this transformation of the constant L_k of (2.3), the constant L_k^* of (2.10), and the constant $L_K^{*\sharp}$ of (2.14), respectively.

The constant L_k of (2.3) does not depend on the values of the constants α_k, hence the above transformation leads to the square of L_k, which is exactly the constant L_k of (2.2), which finishes the proof of Theorem 2.1. The constant L_k^* of (2.10) is a function of $\delta_- = \frac{\alpha_{k-}}{\alpha_k}$ and $\delta_+ = \frac{\alpha_k}{\alpha_{k+}}$. Since replacing all numbers α_i, $i = 0, \ldots, d+1$, by their square roots, reduces to replace the numbers δ_- and δ_+ by their square roots too, the above transformation of the constant L_k^* of (2.10) consists in taking its square and replacing δ_- and δ_+ by their square roots, which leads to the constant L_k of (2.9), which finishes the the proof of Theorem 2.5. Similarly, the above transformation of the constant $L_k^{*\sharp}$ of (2.14) leads to the constant $L_k^\sharp$ of (2.13), which finishes the proof of Theorem 2.6.

Now Theorem 2.4 is an immediate corollary of Theorem 2.5. Indeed as said in Section 2, taking $k^- = 0$, $k^+ = d+1$ in Theorem 2.5, and using $L_k^- \leqslant L_k$, $L_k^{-*} \leqslant L_k^*$ and $\mathcal{U}(k, 0) = \mathcal{E}(k)(k+1)$, we get Theorem 2.4. □

Corollaries 2.8 and 2.9 are straightforward consequences of Theorem 2.5 and Proposition 2.7. We only need to prove the latter proposition.

Proof of Proposition 2.7. Let us use the notations of Theorem 2.5, and let $k_0 = 0, k_1, \ldots, k_q = d$ be the sequence of abscissæ of the vertices of the Newton polygon of $\mathsf{t}p(x)$, as shown in Figure 2.1. Then, the ratio δ_- does not change when k^- moves inside an interval $[k_{r-1} + 1, k_r]$. Since $\mathcal{U}(\cdot, \delta)$ is a nondecreasing function, in order to maximize L_k^- or L_k^{-*} and thus L_k, with δ_- constant (and δ_+ and k^+ constant), we need to minimize $k - k^-$, which implies that $k^- = k_r$ for some $0 \leqslant r \leqslant q$ such that $k_r < k$. Similarly, $k^+ = k_s + 1$ for some $0 \leqslant s \leqslant q$ such that $k_s \geqslant k$. □

REMARK 5.7. All the lower bounds in Theorems 2.1, 2.4, 2.5, and 2.6 are equivalent to inequalities of the form

$$(|\zeta_1 \ldots \zeta_{nk}|^{-1} |\det A_0|)^{1/n} L_k \leqslant \|A_0\| \alpha_1^{-1} \ldots \alpha_k^{-1} . \qquad (5.8)$$

By definition of the tropical roots, $\|A_0\| \alpha_1^{-1} \ldots \alpha_k^{-1} = \exp \hat{p}_k$, where $\hat{p}_k$ is the value in k of the concave hull of the map $j \in \{0, \ldots, d\} \mapsto \log \|A_j\|$. Hence, if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on the space of matrices such that $\|A\| \leqslant \|A\|'$ for all $A \in \mathbb{C}^{n \times n}$, then the right hand side in (5.8) is smaller for $\|\cdot\|$ than for $\|\cdot\|'$. Moreover, in (2.2), (2.3), (2.5), and (2.6), L_k depends only on P or k and n, but not on the norm $\|\cdot\|$, so that (5.8) is necessarily a tighter inequality for $\|\cdot\|$ than for $\|\cdot\|'$. In particular if (A2) holds with Q and Q', then the lower bounds (2.3) and (2.6) for $\|\cdot\|$ are weaker than the corresponding ones for $A \mapsto \|QAQ'\|_{\mathrm{nF}}$. For the lower bounds (2.9), (2.10), (2.13), and (2.14) of Theorems 2.5 and 2.6, and the ones of Corollaries 2.8 and 2.9, based on the separation between the tropical roots, the comparison is not so simple, because changing the norm changes the separation between the tropical roots.

To prove Theorem 2.12, we first prove the following lemma which provides a lower bound for the modulus of the coefficients of the polynomial $\det(P(\lambda))$.

LEMMA 5.8. *Let* $A_0, \ldots, A_d$, P, $\|\cdot\|$, η, $\alpha_1, \ldots, \alpha_d$, q, $\delta_0, \ldots, \delta_q$, *and* $C_{n,d,k}$ *be*

as in Theorem 2.12, and denote by $\tilde{p}$ the polynomial:

$$\tilde{p}(\lambda) = \det P(\lambda) = \sum_{l=0}^{nd} \tilde{p}_l \lambda^l \quad . \tag{5.9}$$

Then, for $j = 0, \ldots, q$, we have $\|A_{k_j}\| > 0$ and

$$|\tilde{p}_{nk_j}| \geqslant |\det A_{k_j}| - (C_{n,d,nk_j} - 1)\eta\|A_{k_j}\|^n \delta_j \quad .$$

Proof. Let $k = 0, \ldots, d$. Using the multilinearity of the determinant we get

$$\tilde{p}(\lambda) = \det \sum_{j=0}^{d} A_j \lambda^j = \sum_{\phi} \det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)})\lambda^{\sum_{m=1}^{n} \phi(m)} \quad ,$$

where the sum is taken over all maps $\phi : \{1, \ldots, n\} \to \{0, \ldots, d\}$. Denoting by Φ_k the set of all such maps that satisfy $\sum_{m=1}^{n} \phi(m) = nk$, we obtain that the nk-th coefficient of the polynomial $\tilde{p}$ is equal to:

$$\tilde{p}_{nk} = \sum_{\phi \in \Phi_k} \det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)})$$

$$= \det A_k + \sum_{\phi \in \Phi_k, \ \phi \not\equiv k} \det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)}) \quad .$$

Using Hadamard's inequality together with the definition of η yields

$$|\det(A_{\phi(1)}^{(1)}, A_{\phi(2)}^{(2)}, \ldots, A_{\phi(n)}^{(n)})| \leqslant \|A_{\phi(1)}^{(1)}\|_2 \ldots \|A_{\phi(n)}^{(n)}\|_2 \leqslant \eta\|A_{\phi(1)}\| \ldots \|A_{\phi(n)}\| \quad .$$

Assume now that $k = k_j$ for some $j = 0, \ldots, q$. Then, k is the abscissa of a vertex of the Newton polygon of the tropical polynomial $\mathsf{t}p$ defined in Theorem 2.1. By Proposition 3.3 applied to $\mathsf{t}p$, we get that $\|A_k\| > 0$, $\|A_m\| \leqslant \|A_k\| \prod_{\ell=m+1}^{k} \alpha_\ell \leqslant \|A_k\|\alpha_k^{k-m}$ for all $m \leqslant k$ and $\|A_m\| \leqslant \|A_k\| \prod_{\ell=k+1}^{m} \alpha_\ell^{-1} \leqslant \|A_k\|\alpha_{k+1}^{k-m} \leqslant \|A_k\|\alpha_k^{k-m}\delta_j^{m-k}$ for all $m > k$. Hence

$$\|A_{\phi(1)}\| \ldots \|A_{\phi(n)}\| \leqslant \|A_k\|^n \alpha_k^{\sum_{m=1}^{n}(k-\phi(m))} \delta_j^{\sum_{m=1,\phi(m)>k}^{n}(\phi(m)-k)}$$

$$= \|A_k\|^n \delta_j^{\sum_{m=1,\phi(m)>k}^{n}(\phi(m)-k)} \quad ,$$

when $\sum_{m=1}^{n} \phi(m) = nk$. When in addition $\phi \not\equiv k$, there exists $m = 1, \ldots, n$ such that $\phi(m) > k$, thus $\sum_{m=1,\phi(m)>k}^{n}(\phi(m) - k) \geqslant 1$, which yields

$$\|A_{\phi(1)}\| \ldots \|A_{\phi(n)}\| \leqslant \|A_k\|^n \delta_j \quad .$$

From all the above inequalities, we deduce that

$$|\tilde{p}_{nk}| \geqslant |\det A_k| - \sum_{\phi \in \Phi_k, \ \phi \not\equiv k} \eta\|A_k\|^n \delta_j \quad .$$

Since by definition $C_{n,d,nk}$ is the cardinality of Φ_k, and there exists exactly one element of Φ_k such that $\phi \equiv k$, we obtain the inequality of the lemma for $k = k_j$. $\square$

Proof of Theorem 2.12. Consider the polynomial $\tilde{p}$ of (5.9) and let $\gamma_1, \ldots, \gamma_{nd}$ be the tropical roots of the tropical polynomial $\mathsf{t}\tilde{p}(x) = \max_{0 \leqslant l \leqslant nd} |\tilde{p}_l| x^l$ arranged in non-decreasing order. Let $j = 0, \ldots, q$ and denote $k = k_j$. From (1.2), we have

$$|\zeta_1 \ldots \zeta_{nk}| \leqslant \binom{nd}{nk} \gamma_1 \ldots \gamma_{nk} \ , \qquad (5.10)$$

By the first part of Proposition 3.3 applied to the tropical polynomial $\mathsf{t}\tilde{p}$, we get $|\tilde{p}_{nk}|\gamma_1 \ldots \gamma_{nk} \leqslant |\tilde{p}_0|$. Applying the result of Lemma 5.8, and using the assumption of Theorem 2.12 on c_j, we get that $\tilde{p}_{nk} \geqslant c_j \|A_k\|^n > 0$. Since $\tilde{p}_0 = |\det A_0|$, we deduce that $\gamma_1 \ldots \gamma_{nk} \leqslant |\det A_0|/(c_j \|A_k\|^n)$. Gathering this with (5.10), we get

$$|\zeta_1 \ldots \zeta_{nk}| \leqslant \frac{\binom{nd}{nk}}{c_j} \frac{|\det A_0|}{\|A_0\|^n} \frac{\|A_0\|^n}{\|A_k\|^n} \ .$$

Since $k = k_j$ is the abscissa of a vertex of the Newton polygon of the tropical polynomial $\mathsf{t}p$, we get from Proposition 3.3 applied to $\mathsf{t}p$ that $\frac{\|A_0\|}{\|A_k\|} = \alpha_1 \ldots \alpha_k$, hence the previous inequality shows Theorem 2.12. $\square$

6. Examples. In this section we provide some examples to show that the lower bounds stated in Section 2 can be tight when the tropical roots are well separated and when the input matrices are well conditioned. We used the tropical scaling algorithm introduced in [GS09] to compute the eigenvalues of a given matrix polynomial. We warn the reader that a naive double precision numerical computation (linearization, followed by QZ algorithm) fails to give accurate values for some eigenvalues, because eigenvalues have different orders of magnitude (hence, either adapted scalings, or extended precision arithmetic, must be used). All the computations were performed in Scilab 5.3.0.

In the following tables, we are comparing the value of the ratio $\frac{|\zeta_1 \ldots \zeta_{nk}|}{(\alpha_1 \ldots \alpha_k)^n}$ with the lower bounds (2.1) of Section 2 for some matrix polynomials P. In view of Remark 5.7, we computed only the lower bounds obtained under (A2) for the normalized Frobenius norm $\| \cdot \|_{\mathrm{nF}}$, and the lower bounds obtained under (A1) for the normalized Schatten 1-norm (or normalized trace norm) $\| \cdot \|_{*1}$ (which satisfies (A1) but not (A2), see Property 5.4). Then, in each table, for each possible value of $k = 1, \ldots, d$, the column RATIO gives $\frac{|\zeta_1 \ldots \zeta_{nk}|}{(\alpha_1 \ldots \alpha_k)^n}$, and the other columns give the lower bound cL_k^n of (2.1) with different values of L_k. The column G-PÓLYA gives the universal lower bound, generalizing the lower bound of Pólya to the matrix case, for which L_k is given in (2.5) or (2.6), the column FEWNOM gives the lower bound involving the number of monomials, for which L_k is given in (2.2) or (2.3), and the column SEPAR gives the lower bound based on the separation between the tropical roots, for which L_k is replaced by the constant L_k^{prox} of (2.18) or $L_k^{*,\mathrm{prox}}$ of (2.19).

EXAMPLE 6.1. Consider the following matrix polynomial

$$P_1(\lambda) = 10^{-7} U_0 + 10^2 \lambda^2 U_2 + 10^7 \lambda^4 U_4 + 10 \lambda^7 U_7 + 10^{-8} \lambda^9 U_9 \ ,$$

where all the matrices U_j, $j \in \{0, 2, 4, 7, 9\}$, are unitary of dimension 3. Here, we shall only consider the normalized Frobenius norm since for a unitary matrix U, we have $\|U\|_{\mathrm{nF}} = \|U\|_{*1} = 1$, so the tropical polynomial associated to P_1 are the same for both norms, and the lower bounds under (A1) are the same, and are thus weaker than the lower bounds under (A2) for the normalized Frobenius norm. The Newton polygon of the tropical polynomial corresponding to P_1 (for the normalized Frobenius
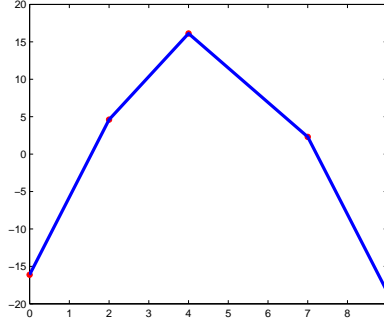
FIG. 6.1. *The Newton polygon of the tropical polynomial corresponding to P_1.*

norm) is shown in Figure 6.1, and its tropical roots are $10^{-9/2}$, $10^{-5/2}$, 10^2, and $10^{9/2}$ with multiplicities 2, 2, 3, and 2, respectively. Table 6.1 shows the lower bounds for all values of $1 \leqslant k \leqslant 9$, and the maximum of the ratios for a sample of 1000 random choices of the unitary matrices U_j. The ratios slightly change for different random choices of the unitary matrices U_j, but with a difference smaller than 5.910^{-4}. Note that when $1 \leqslant k \leqslant 9$ increases, the generalized lower bound of Pólya decreases, and should be replaced by its symmetrized version with k replaced by $d - k$. However the bounds using the separation between tropical roots are the best ones, and they are tight at the vertices of the Newton polygon of the tropical polynomial associated to P_1.

k	RATIO	FEWNOM	G-PÓLYA	SEPAR
1	1.00015	0.08944	0.125	0.125
2*	1.00029	"	0.05702	0.97044
3	1.00015	"	0.03425	0.125
4*	1	"	0.02345	0.99991
5	1.00000	"	0.01733	0.125
6	1.00000	"	0.01348	0.12500
7*	1	"	0.01088	0.99056
8	1	"	0.00901	0.125
9*	1	"	0.00763	1

TABLE 6.1

Comparison of the bounds for P_1 using the normalized Frobenius norm. The abscissæ of the vertices of the Newton polygon of corresponding tropical polynomial are indicated by the symbol *.

EXAMPLE 6.2. In the following example, we perturb the previous polynomial by adding coefficients to P_1, in such a way that the Newton polygon of the tropical polynomial associated to P_1 does not change, either for the normalized Frobenius norm or at least for the normalized Schatten 1-norm. Here we fix the unitary matrices to be either the identity matrix or its opposite, and we consider the following matrices:

$$B = b \begin{pmatrix} 10^{-15} & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}, \quad C = \frac{\sqrt{3}}{14} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

where b is chosen so that $\|B\|_{\mathrm{nF}} = 1$, and C is also such that $\|C\|_{\mathrm{nF}}=1$. The matrix C is singular, since it has rank 1, and it satisfies $\|C\|_{*1} = 1/\sqrt{3}$. The matrix B is ill-conditioned, so that $\|B\|_{*1} < \|B\|_{\mathrm{nF}} = 1$ too.

We shall consider the two polynomials:

$$P_2(\lambda) = 10^{-7}U_0 + 10^{-5/2}\lambda C_1 + 10^2\lambda^2 U_2 + 10^{9/2}\lambda^3 C_1 + 10^7\lambda^4 U_4 + 10^5\lambda^5 C_1$$
$$+ 10^3\lambda^6 B_1 + 10\lambda^7 U_7 + 10^{-7/2}\lambda^8 B_1 + 10^{-8}\lambda^9 U_9$$
$$P_3(\lambda) = 10^{-7}U_0 + 10^{-5/2}\lambda C_2 + 10^2\lambda^2 U_2 + 10^{9/2}\lambda^3 C_2 + 10^7\lambda^4 U_4 + 10^5\lambda^5 C_2$$
$$+ 10^3\lambda^6 B_2 + 10\lambda^7 U_7 + 10^{-7/2}\lambda^8 B_2 + 10^{-8}\lambda^9 U_9 \ ,$$

with $U_0 = U_4 = U_7 = -I$, $U_2 = U_9 = I$, $B_1 = B$, $C_1 = C$, $B_2 = \|B\|_{*1}^{-1}B$, $C_2 = \sqrt{3}C$. Since for any unitary matrix U with dimension 3, $\|U\|_{*1} = \|U\|_{\mathrm{nF}} = 1$, and since for any complex matrix A of dimension 3, we have $\|A\|_{*1} \leqslant \|A\|_{\mathrm{nF}}$, the tropical polynomial associated to P_2 for either the normalized Frobenius norm or the Schatten 1-norm coincides with the one associated to P_1. Hence, as for P_1, the bounds for P_2 with the normalized Schatten 1-norm are necessarily weaker than the ones with the normalized Frobenius norm. Moreover, all the bounds presented in Table 6.1 for P_1 remain the same for P_2 except the bounds based on the number of nonzero coefficients. We present all these bounds in Table 6.2 together with the new ratios. Since these ratios are different from the ones of Table 6.1, one can see that the lower bounds based on the separation between tropical roots may be tight.

k	RATIO	FEWNOM	G-PÓLYA	SEPAR
1	0.45628	0.03162	0.125	0.125
2*	1.03298	"	0.05702	0.97044
3	0.45627	"	0.03425	0.125
4*	1.00009	"	0.02345	0.99991
5	0.39258	"	0.01733	0.125
6	0.39258	"	0.01348	0.12500
7*	1.00867	"	0.01088	0.99056
8	0.47428	"	0.00901	0.125
9*	1	"	0.00763	1

TABLE 6.2

Comparison of the bounds for P_2 using the normalized Frobenius norm. The abscissæ of the vertices of the Newton polygon of the corresponding tropical polynomial are indicated by the symbol *.

Let us consider now the polynomial P_3. The normalized Schatten 1-norm of the coefficients of P_3 coincide with the normalized Frobenius norm of the coefficients of P_2, hence the tropical polynomial associated to P_3 for the normalized Schatten 1-norm coincides with the one associated to P_1 for the normalized Frobenius norm, so that the tropical roots are the same. However, since the Schatten 1-norm satisfies (A1) but not (A2), we can only get the bounds based on (A1) which are necessarily lower than the one presented above for P_1 with the normalized Frobenius norm. Finally, the tropical polynomial associated to P_3 for the normalized Frobenius norm differs from the one associated to P_1, so that the tropical roots and the bounds differ too. All tropical roots of this new tropical polynomial have multiplicity 1, which means that all indices are abscissæ of vertices of its Newton polygon. We present in Table 6.3, the ratios and the lower bounds for both normalized Frobenius and Schatten 1-norms. Here the ratios and lower bounds for the normalized Schatten 1-norm are lower than the ones for the normalized Frobenius norm. However, one can see that the results for the normalized Frobenius norm are still better than the ones for the normalized Schatten 1-norm.

EXAMPLE 6.3. In order to obtain a polynomial for which the lower bounds for the normalized Schatten 1-norm are better than the ones for the normalized Frobenius

	norm. Frobenius norm			norm. Schatten 1-norm			
k	RATIO	FEWNOM	G-PÓLYA	SEPAR	RATIO	FEWNOM	G-PÓLYA	SEPAR
1	1.57223	0.03162	0.125	0.35355	0.30258	0.001	0.01563	0.01563
2*	1.11466	"	0.05702	0.91391	1.11466	"	0.00325	0.54771
3	1.57223	"	0.03425	0.35355	0.30258	"	0.00117	0.01562
4*	1.00028	"	0.02345	0.99972	1.00028	"	0.00055	0.96682
5	1.45041	"	0.01733	0.18345	0.27913	"	0.00030	0.01562
6	1.01082	"	0.01348	0.05989	0.27913	"	0.00018	0.01533
7*	1.02104	"	0.01088	0.97788	1.02104	"	0.00012	0.71337
8	1.26686	"	0.00901	0.25721	0.34983	"	0.00008	0.01563
9*	1	"	0.00763	1	1	"	0.00006	1

TABLE 6.3

Comparison of the bounds for P_3 using the normalized Frobenius and Schatten 1-norms. The abscissæ of the vertices of the Newton polygon of the tropical polynomial associated to P_3 for the normalized Frobenius norm are indicated by the symbol *.

norm, one need to take matrices A_i such that $\|A_i\|_{\mathrm{nF}}/\|A_i\|_{*1}$ is as large as possible. This means that the dimension n is large and that some of the singular values of A_i are zero, or at least that the ratio between the maximal and minimal singular values is large, which implies that A_i is singular or at least ill-conditioned. Moreover some parameters of the lower bounds need to be smaller than $\|A_i\|_{\mathrm{nF}}/\|A_i\|_{*1}$, which is possible for instance for the lower bounds based on the separation between tropical roots, when these tropical roots have a small multiplicity and are well separated. We shall show here such examples.

We consider a matrix polynomial with degree 3 and dimension $n = 10$:

$$P_4(\lambda) = A_0 + A_1\lambda + A_2\lambda^2 + A_3\lambda^3 \ ,$$

where all the matrices $A_i, i = 0, \ldots, 3$ are diagonal matrices: $A_i = \mathrm{diag}(d_i)$ with $d_i \in \mathbb{C}^n$. We choose $A_0 = A_3 = I$, so that $c = 1$, $\alpha_1\alpha_2\alpha_3 = 1$ (for norms such that $\|I\| = 1$) and for $k = 3$ the ratio $\frac{|\zeta_1 \ldots \zeta_{nk}|}{(\alpha_1 \ldots \alpha_k)^n}$ is equal to 1, which is equal to the lower bound based on separation of tropical roots. We choose the diagonal elements of the matrices A_1, A_2 as follows:

$$d_1 := \frac{a_1}{1 + b/n}(1, \ldots, 1, 1 + b) \ ,$$

$$d_2 := \frac{a_2}{1 + b/n}(1 + b, 1, \ldots, 1) \ ,$$

with $a_1, a_2, b > 0$. Then

$$\|A_1\|_{*1} = \frac{\|d_1\|_1}{n} = a_1 \ ,$$

$$\|A_2\|_{*1} = \frac{\|d_2\|_1}{n} = a_2 \ ,$$

and

$$\|A_1\|_{\mathrm{nF}} = \frac{\|d_1\|_2}{\sqrt{n}} = a_1 f_b \ ,$$

$$\|A_2\|_{\mathrm{nF}} = \frac{\|d_2\|_2}{\sqrt{n}} = a_2 f_b \ ,$$

with

$$f_b := \frac{\sqrt{1 + \frac{2b + b^2}{n}}}{1 + b/n} > 1 \ .$$

The constant f_b is large when b is large, that is when the condition numbers of A_1 and A_2 are large.

When $a_1 = a_2 > 1$, the tropical polynomial associated to P_4 for the normalized Schatten 1-norm (resp. the normalized Frobenius norm) has 3 different tropical roots equal to $1/a_1$, 1 and a_1 (resp. $1/(a_1 f_b)$, 1 and $a_1 f_b$). We present in Tables 6.4 and 6.5, the ratios and the lower bounds for both normalized Frobenius and Schatten 1-norms, when $a_1 = 10^4$, and $b = 2$ and $b = 10$ respectively. When $b = 10$, the matrices are ill-conditioned, so that the ratios $\frac{|\zeta_1...\zeta_{nk}|}{(\alpha_1...\alpha_k)^n}$ for $k = 1$ or 2 are very large. When either $b = 2$ or $b = 10$, we see that the lower bounds based on the separation between the tropical roots are the best ones, and that they are nearer the ratios $(\frac{|\zeta_1...\zeta_{nk}|}{(\alpha_1...\alpha_k)^n})$ in the case of the Schatten 1-norm than in the case of the Frobenius norm, although they are far in the case where $b = 10$.

	norm. Frobenius norm				norm. Schatten 1-norm			
k	RATIO	FEWNOM	G-PÓLYA	SEPAR	RATIO	FEWNOM	G-PÓLYA	SEPAR
1*	6.30697	0.00098	0.00098	0.99911	2.06667	9.5D-07	9.5D-07	0.81873
2*	6.30697	0.00098	0.00007	0.99911	2.06667	9.5D-07	5.1D-09	0.81873
3*	1	0.00098	0.00001	1	1	9.5D-07	1.7D-10	1

TABLE 6.4

*Comparison of the bounds for P_4 with $a_1 = a_2 = 10^4$ and $b = 2$. The abscissæ of the vertices of the Newton polygon of corresponding tropical polynomial are indicated by the symbol *.*

	norm. Frobenius norm				norm. Schatten 1-norm			
k	RATIO	FEWNOM	G-PÓLYA	SEPAR	RATIO	FEWNOM	G-PÓLYA	SEPAR
1*	33882.7	0.00098	0.00098	0.99945	93.4462	9.5D-07	9.5D-07	0.81873
2*	33882.7	0.00098	0.00007	0.99945	93.4462	9.5D-07	5.1D-09	0.81873
3*	1	0.00098	0.00001	1	1	9.5D-07	1.7D-10	1

TABLE 6.5

*Comparison of the bounds for P_4 with $a_1 = a_2 = 10^4$ and $b = 10$. The abscissæ of the vertices of the Newton polygon of corresponding tropical polynomial are indicated by the symbol *.*

When $a_1 > 1$ and $a_2 = \sqrt{a_1/f_b}$ the tropical polynomial associated to P_4 for the normalized Schatten 1-norm (resp. the normalized Frobenius norm) has only 2 different tropical roots equal to $1/a_1$ and $\sqrt{a_1}$ (resp. $1/(a_1 f_b)$, 1 and $\sqrt{a_1 f_b}$) with respective multiplicities 1 and 2. We present in Tables 6.6 and 6.7, the ratios and the lower bounds for both normalized Frobenius and Schatten 1-norms, when $a_1 = 10^4$, and $b = 2$ and $b = 10$ respectively. When $b = 10$, the matrices are ill-conditioned, so that for $k = 1$ or 2 the distances (ratios) between the ratio $\frac{|\zeta_1...\zeta_{nk}|}{(\alpha_1...\alpha_k)^n}$ and its lower bounds are very large, although for $k = 2$ this ratio is smaller than in the case where $a_1 = a_2 = 10^4$ above. When either $b = 2$ or $b = 10$, we see that the lower bounds based on the separation between the tropical roots are the best ones. For $k = 2$ which is the abscissa of a vertex of the Newton polygon, the lower bound obtained in the case of the Schatten 1-norm is nearer the ratio $\frac{|\zeta_1...\zeta_{nk}|}{(\alpha_1...\alpha_k)^n}$ than the one obtained in the case of the Frobenius norm, whereas for $k = 3$ the contrary holds.

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	norm. Frobenius norm				norm. Schatten 1-norm			
k	RATIO	FEWNOM	G-PÓLYA	SEPAR	RATIO	FEWNOM	G-PÓLYA	SEPAR
1*	6.29864	0.00098	0.00098	0.99999	2.06394	9.5D-07	9.5D-07	0.98020
2	0.68418	0.00098	0.00007	0.00098	0.39165	9.5D-07	5.1D-09	9.5D-07
3*	1	0.00098	0.00001	1	1	9.5D-07	1.7D-10	1

TABLE 6.6

*Comparison of the bounds for P_4 with $a_1 = 10^4$, $a_2 = \sqrt{a_1/f_b}$, and $b = 2$. The abscissæ of the vertices of the Newton polygon of corresponding tropical polynomial are indicated by the symbol *.*

	norm. Frobenius norm				norm. Schatten 1-norm			
k	RATIO	FEWNOM	G-PÓLYA	SEPAR	RATIO	FEWNOM	G-PÓLYA	SEPAR
1*	33754.9	0.00098	0.00098	1.00000	93.0935	9.5D-07	9.5D-07	0.98020
2	9.86557	0.00098	0.00007	0.00098	0.5181	9.5D-07	5.1D-09	9.5D-07
3*	1	0.00098	0.00001	1	1	9.5D-07	1.7D-10	1

TABLE 6.7

*Comparison of the bounds for P_4 with $a_1 = 10^4$, $a_2 = \sqrt{a_1/f_b}$, and $b = 10$. The abscissæ of the vertices of the Newton polygon of corresponding tropical polynomial are indicated by the symbol *.*

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